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Mean-motion resonances in satellite–disc interactions

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ABSTRACT
Mean-motion resonances between a Keplerian disc and an orbiting companion are analysed within a Hamiltonian formulation using complex canonical Poincaré variables, which are ideally suited to the description of eccentricity and inclination dynamics. Irreversibility is introduced by allowing for dissipation within the disc. A method is given for determining the rates of change of eccentricity and inclination variables of the disc and companion associated with resonances of various orders, including both reversible and irreversible effects, which extend and generalize previous results. Preliminary applications to protoplanetary systems and close binary stars are discussed.

Key words: accretion, accretion discs — binaries: close — celestial mechanics — planets: rings — planets and satellites: general

1 INTRODUCTION

1.1 Satellite–disc interactions

The gravitational interaction of a gaseous or particulate disc with an orbiting companion is of general interest in astrophysics. Important examples include protoplanetary systems involving one or more planets orbiting within a gaseous disc, accretion discs in close binary stars, and systems of planetary rings and moons.

As in the planetary theory of classical celestial mechanics, the disc and companion experience deviations from perfect Keplerian motion around the central mass as a result of their mutual gravitational perturbations. Sufficiently close to the companion's orbit, the behaviour of the disc can be complicated and highly nonlinear, leading in many cases to an exclusion of the disc from the coorbital region. The extent of such a gap depends on the ratio of the mass of the companion to that of the central object, and on the properties of the disc. Away from this region, the interaction can be analysed by perturbation methods and has two principal forms.

Secular interactions (of first order in the perturbation) arise from the mass distribution of the companion averaged around its orbit. These contribute to the precessional behaviour of the system and allow slow, reversible exchanges of eccentricity and inclination between the companion and the disc. Such behaviour is analogous to the Laplace–Lagrange secular theory of planetary systems in which, for example, Jupiter and Saturn undergo large-amplitude oscillatory exchanges of eccentricity and inclination over tens of thousands of years (e.g. Murray & Dermott 1999).

Mean-motion resonances depend specifically on the periodic nature of the companion's orbit. The fluctuating forces induced by the companion generally give rise to rapid oscillations of small amplitude in the disc that are of little consequence. However, where a commensurability occurs between the orbital periods of the companion and a particle in the disc, a strong localized interaction occurs. Through the intervention of collective effects and dissipation in the disc, a different kind of secular behaviour is induced at second order in the perturbation. This leads to irreversible changes in semimajor axis, eccentricity and inclination of the disc and companion, rather than just precessional behaviour.

1.2 Corotation and Lindblad resonances

Mean-motion resonances in satellite–disc interactions were studied in detail in an influential series of papers by Goldreich & Tremaine. Early in this series (Goldreich & Tremaine 1978, 1979, 1980) the authors considered a two-dimensional problem in which a circular disc is perturbed by a companion with a slightly eccentric orbit. To calculate the linear response of the disc, they decomposed the perturbing potential by Fourier analysis into a series of rigidly rotating components with various amplitudes, azimuthal wavenumbers and angular pattern speeds. In a system with Keplerian orbits, the angular pattern speeds that occur are rational multiples p/q of the mean motion of the companion, the amplitude scaling with the $|p-q|$th power of the eccentricity.

The linearized equations governing the response of the disc at a given azimuthal wavenumber and angular pattern speed allow for the propagation of free waves in certain intervals of radius. Three important radii are the corotation radius, where the wave frequency as measured in a frame rotating locally with the disc vanishes, and the two Lindblad radii, where the same quantity equals (plus or minus) the epicyclic frequency. In the simplest case of a two-dimensional non-self-gravitating disc, only one wave mode is permitted. This 'density wave' combines inertial (epicyclic) and acoustic behaviour and can propagate only where the wave frequency seen by the disc exceeds the epicyclic frequency in magnitude. It therefore propagates interior to the inner Lindblad radius and exterior to the outer Lindblad radius, but is evanescent between the Lindblad radii where the corotation radius lies.

The forcing of disturbances by a perturbing potential component is effectively localized to the neighbourhoods of the corotation and Lindblad radii. Beyond the Lindblad radii, the forcing is ineffective because of the very limited overlap between the potential and the free waves, which oscillate rapidly in radius. Despite the evanescent nature of the free waves near the corotation radius, it plays an important role as a singularity of the linearized equations. The forcing can then be said to occur at corotation and Lindblad *resonances*, which can be identified with the mean-motion resonances of celestial mechanics. The reason for this is that, in a Keplerian system where the epicyclic frequency equals the orbital frequency, any radius where the orbital frequency is commensurate with the mean motion of the companion serves, in principle, as both corotation and Lindblad resonances for some potential components.

At a *corotation resonance* the linearized response of the disc at first order in the perturbation is singular but can be resolved by including a viscosity. At second order in the perturbation, the companion exerts a secular torque on the disc as it generates a localized, non-wavelike disturbance that transfers angular momentum steadily to the disc through viscous stresses.

At a *Lindblad resonance* the localized response is regularized by collective effects such as pressure, self-gravity or viscosity, and is generally in the form of an attenuated wave propagating away from the resonance. At second order in the perturbation, and provided that the wave is dissipated somehow within the disc, the companion again exerts a secular torque in launching the disturbance (see also Meyer-Vernet & Sicardy 1987).

In a three-dimensional disc, not considered by Goldreich & Tremaine (1979), a *vertical resonance* is similar to a Lindblad resonance but involves forcing and motion perpendicular to the plane of the disc, so that a bending wave is excited rather than a density wave.

Convenient formulae are available for the corotation and Lindblad torques, which have an important role in determining the orbital evolution of the companion and the surface density distribution of the disc. The subsequent literature has made extensive use of these expressions, as well as providing some modifications to the torque formulae (Ward 1989; Artymowicz 1993; Ogilvie & Lubow 2003). Together, these results provide the basis of our understanding of planet–disc and other satellite–disc interactions.

1.3 Alternative treatments of mean-motion resonances

It is perhaps not widely appreciated that the approach described above is inadequate for general satellite–disc interactions. A knowledge of the resonant torque (by which is meant the component of the torque perpendicular to the disc) determines one component of the rates of change of the orbital angular momentum vectors of the disc and companion and also, by virtue of Jacobi's theorem, the rate of change of orbital energy of the companion; however, this information does not generally suffice to determine the secular rates of change of the relevant orbital elements of the disc and companion. Furthermore, the method does not apply to the general case in which the disc is eccentric, because the forced wave equations in an eccentric disc have not been analysed.

Later in their series of papers, Goldreich & Tremaine (1981) and Borderies, Goldreich & Tremaine (1984) employed methods of celestial mechanics to treat cases in which either the companion or the disc has an eccentric and/or inclined orbit. Using 'simple artifices' they derived formulae for the rates of change of the canonical action variables associated with mean-motion resonances, in terms of the classical disturbing function, while avoiding an explicit discussion of collective effects. In principle, these expressions are considerably more powerful and general than the earlier torque formulae, yet they appear not to have been used in the subsequent literature. We have tried to make use of them in connection with eccentric planet–disc interactions and similar problems but found that they generally give results that are not of the desired form.

One aspect of their method is that it isolates, in the traditional way, terms of different angular arguments in the disturbing function. An implicit averaging is therefore involved over the mutual apsidal and nodal precession of the disc and companion. At least in some contexts, this procedure is inappropriate because the mutual precession may occur on a timescale comparable to that of the secular evolution induced by mean-motion resonances, or indeed may not occur at all if the disc and companion are engaged in a secular normal mode (e.g. Lubow & Ogilvie 2001). The formulae therefore describe the rates of change of a,

e and i averaged over the timescales of mutual precession, while what are generally desired are the rates of change of a, e, i, ϖ and Ω averaged only over the relative mean motion.[1]

A second aspect is the way that dissipation is introduced into the analysis, which is essential to capture the irreversible nature of the effects of mean-motion resonances. This is done either by including ad hoc damping terms (Goldreich & Tremaine 1981) or is avoided altogether by using Landau's prescription in which the disturbing function is given a slow exponential growth (Borderies, Goldreich & Tremaine 1984). Although mathematically convenient, these approaches can give misleading results. The effects of mean-motion resonances depend on the fact that the disc conserves angular momentum while it dissipates energy, and the damping terms added to the equations ought to respect these properties. The Landau approach is problematic because the first-order oscillatory disturbances in the disc are permitted to grow exponentially rather than being steadily dissipated, making it difficult to calculate the second-order secular changes correctly.

The purpose of the present paper is to revisit the analysis of mean-motion resonances in satellite–disc interactions and to derive useful and general formulae for the associated evolution of orbital elements of the disc and companion. We employ methods of celestial mechanics and treat the collective effects of the disc in an approximate way that is adequate for our present purposes. The remainder of the paper is organized as follows. In Section 2 we consider the restricted problem in which the companion has a prescribed orbit. We develop the theory using a Hamiltonian formulation of the equations, and describe the role of collective effects. In Section 3 we consider the coupled satellite–disc system consistently and determine the feedback of mean-motion resonances on the companion. Preliminary applications are worked out in Section 4 and a summary of the main ideas and results follows in Section 5.

2 THE RESTRICTED PROBLEM

2.1 Hamiltonian formulation

We consider a thin disc around a central mass M, and initially neglect any collective effects. The disc is perturbed by an orbiting companion of mass M'. In the *restricted problem* the perturber has a prescribed periodic orbit that is unaffected by the disc. This approximation is usually applicable, for example, to accretion discs in binary stars. In this case the influence of the companion on a test particle in the disc can be described through a disturbing function that is a specified function of the position of the particle and is also a periodic function of time.

In order to treat the dynamical equations in the most efficient way, we adopt a Hamiltonian formulation using canonical variables (e.g. Morbidelli 2002). The mean longitude λ is conjugate to the energy-related action variable $\Lambda = (GMa)^{1/2}$, where a is the semimajor axis. The other two generalized coordinates and conjugate momenta, which relate to eccentricity and inclination and need not be in action–angle form, are written as q_α and p_α, with $\alpha = \{1, 2\}$. Hamilton's equations for a test particle are

$$\dot{\lambda} = \frac{\partial H}{\partial \Lambda}, \qquad \dot{\Lambda} = -\frac{\partial H}{\partial \lambda}, \qquad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \tag{1}$$

where H is the Hamiltonian per unit mass.[2] The unperturbed (Keplerian) Hamiltonian

$$H_0 = -\frac{(GM)^2}{2\Lambda^2} = -\frac{GM}{2a} \tag{2}$$

gives rise to a mean motion

$$\dot{\lambda} = n(\Lambda) = \frac{\mathrm{d}H_0}{\mathrm{d}\Lambda} = \frac{(GM)^2}{\Lambda^3} = \left(\frac{GM}{a^3}\right)^{1/2}. \tag{3}$$

The disturbing function is periodic in the mean longitudes λ and λ' of the test particle and the perturber, and can therefore be expanded in a double Fourier series. We consider a Hamiltonian of the form

$$H = H_0(\Lambda) - \epsilon \,\mathrm{Re}\left[R(\Lambda, q_\alpha, p_\alpha)\,\mathrm{e}^{\mathrm{i}\phi}\right], \tag{4}$$

where $\epsilon = 1$ is a bookkeeping parameter used to identify effects of various orders in the perturbation, R is a complex potential amplitude (proportional to M') and $\phi = k\lambda + k'\lambda'$ is a potentially resonant angle with integer coefficients k and k'.[3] In the restricted problem $\lambda' = n't + \epsilon'$ is a prescribed linear function of time. The double Fourier series in fact gives rise to an infinite number of perturbing terms of this form, which might be labelled using the notation $R_{k,k'}$ and $\phi_{k,k'}$. We may consider

[1] We use the standard notation for the Keplerian orbital elements (e.g. Murray & Dermott 1999): a, e, i, ϖ and Ω denote the semimajor axis, eccentricity, inclination, longitude of pericentre and longitude of ascending node, while λ is the mean longitude. The secular contribution to the evolution of λ is of less interest here.

[2] The true Hamiltonian of the system is $\int H\,\mathrm{d}m$, where $\mathrm{d}m$ is a mass element of the disc.

[3] Note that it is permissible to replace R with R^* if the signs of k and k' are both changed.

them separately because their second-order secular effects will sum in quadrature owing to the phase relations between them. Hamilton's equations are then

$$\dot{\lambda} = n(\Lambda) - \epsilon \,\mathrm{Re}\left[\frac{\partial R}{\partial \Lambda}\,\mathrm{e}^{\mathrm{i}\phi}\right], \tag{5}$$

$$\dot{\Lambda} = \epsilon \,\mathrm{Re}\left[\mathrm{i}kR\,\mathrm{e}^{\mathrm{i}\phi}\right], \tag{6}$$

$$\dot{q}_\alpha = -\epsilon \,\mathrm{Re}\left[\frac{\partial R}{\partial p_\alpha}\,\mathrm{e}^{\mathrm{i}\phi}\right], \tag{7}$$

$$\dot{p}_\alpha = \epsilon \,\mathrm{Re}\left[\frac{\partial R}{\partial q_\alpha}\,\mathrm{e}^{\mathrm{i}\phi}\right]. \tag{8}$$

An important point is that the apsidal and nodal precessional longitudes ϖ and Ω appear in R (through its dependence on q_α and p_α) and not in the exponential. Terms in the disturbing function with the same values of k and k' but different dependences on ϖ and Ω are treated together, not separately. To isolate such terms would be appropriate only if we were to average the behaviour of the system over the mutual apsidal and nodal precession of the disc and companion.

2.2 Perturbative expansion to second order

We expand the dynamical variables in the form $\Lambda = \Lambda_0 + \epsilon\Lambda_1 + \epsilon^2\Lambda_2 + \cdots$, etc., with the intention that Λ_0 describes the unperturbed disc, Λ_1 the periodic oscillation induced by the perturbation at first order, and Λ_2 contains second-order variations. The effects that we are looking for will appear as secular changes of Λ_2, etc.[4]

At leading order all variables are independent of time except for

$$\dot{\lambda}_0 = n(\Lambda_0) = n_0, \tag{9}$$

which describes the unperturbed mean motion of the disc. It is convenient to use either the unperturbed semimajor axis a_0 or the related action variable $\Lambda_0 = (GMa_0)^{1/2}$ to label the orbits in the disc. Then

$$\dot{\phi}_0 = kn_0 + k'n' \tag{10}$$

is a constant depending on Λ_0. By definition, it vanishes at the location of the mean-motion resonance where $n_0/n' = -k'/k$ is a certain positive rational number.

At first order we find from Hamilton's equations

$$\dot{\lambda}_1 = \frac{\mathrm{d}n}{\mathrm{d}\Lambda}\Lambda_1 - \mathrm{Re}\left[\frac{\partial R}{\partial \Lambda}\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{11}$$

$$\dot{\Lambda}_1 = \mathrm{Re}\left[\mathrm{i}kR\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{12}$$

$$\dot{q}_{\alpha 1} = -\mathrm{Re}\left[\frac{\partial R}{\partial p_\alpha}\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{13}$$

$$\dot{p}_{\alpha 1} = \mathrm{Re}\left[\frac{\partial R}{\partial q_\alpha}\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{14}$$

where all terms on the right-hand sides are evaluated on the unperturbed solution, i.e. at $\Lambda = \Lambda_0$, $q_\alpha = q_{\alpha 0}$, $p_\alpha = p_{\alpha 0}$. The solution of equations (12)–(14) is

$$\Lambda_1 = \mathrm{Re}\left[fkR\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{15}$$

$$q_{\alpha 1} = \mathrm{Re}\left[\mathrm{i}f\frac{\partial R}{\partial p_\alpha}\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{16}$$

$$p_{\alpha 1} = \mathrm{Re}\left[-\mathrm{i}f\frac{\partial R}{\partial q_\alpha}\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{17}$$

where

$$f = \frac{1}{\dot{\phi}_0}. \tag{18}$$

[4] This form of perturbative expansion is valid only for a limited time interval, because Λ_2 will eventually become comparable to Λ_0. A more sophisticated approach would be to use the method of multiple timescales, which would yield a uniformly asymptotic solution. However, the notationally simpler approach taken here allows us to calculate the secular rates of change correctly.

Equation (11) then becomes

$$\dot{\lambda}_1 = \mathrm{Re}\left[\frac{\mathrm{d}n}{\mathrm{d}\Lambda} fkR\,\mathrm{e}^{\mathrm{i}\phi_0} - \frac{\partial R}{\partial \Lambda}\,\mathrm{e}^{\mathrm{i}\phi_0}\right] = \mathrm{Re}\left[-f^{-1}\frac{\partial}{\partial\Lambda}(fR)\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{19}$$

for which the solution is

$$\lambda_1 = \mathrm{Re}\left[\mathrm{i}\frac{\partial}{\partial\Lambda}(fR)\,\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{20}$$

and then we find $\phi_1 = k\lambda_1$. To this order, therefore, the variables undergo forced oscillations around their leading-order values with angular frequency $\dot{\phi}_0$. Four arbitrary constants could be included in the solution, corresponding to small shifts in the mean values of the variables, but these are best absorbed into the leading-order terms. The amplitude of the oscillations diverges at the resonance where $\dot{\phi}_0 = 0$.

At second order the equation for $\dot{\lambda}_2$ is not required, but we have also

$$\dot{\Lambda}_2 = \mathrm{Re}\left[\mathrm{i}k\left(\frac{\partial R}{\partial\Lambda}\Lambda_1 + \frac{\partial R}{\partial q_\beta}q_{\beta 1} + \frac{\partial R}{\partial p_\beta}p_{\beta 1} + \mathrm{i}\phi_1 R\right)\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{21}$$

$$\dot{q}_{\alpha 2} = -\mathrm{Re}\left[\left(\frac{\partial^2 R}{\partial p_\alpha \partial\Lambda}\Lambda_1 + \frac{\partial^2 R}{\partial p_\alpha \partial q_\beta}q_{\beta 1} + \frac{\partial^2 R}{\partial p_\alpha \partial p_\beta}p_{\beta 1} + \mathrm{i}\phi_1\frac{\partial R}{\partial p_\alpha}\right)\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{22}$$

$$\dot{p}_{\alpha 2} = \mathrm{Re}\left[\left(\frac{\partial^2 R}{\partial q_\alpha \partial\Lambda}\Lambda_1 + \frac{\partial^2 R}{\partial q_\alpha \partial q_\beta}q_{\beta 1} + \frac{\partial^2 R}{\partial q_\alpha \partial p_\beta}p_{\beta 1} + \mathrm{i}\phi_1\frac{\partial R}{\partial q_\alpha}\right)\mathrm{e}^{\mathrm{i}\phi_0}\right], \tag{23}$$

where, again, terms on the right-hand sides are evaluated on the unperturbed solution, and we adopt the convention that *a summation over* $\beta = \{1, 2\}$ *will be implied wherever it appears*. The right-hand sides, involving products of oscillating quantities, contain both non-oscillatory terms of zero frequency and oscillatory terms of frequency $2\dot{\phi}_0$. We are interested in the secular rates of change of Λ, q_α and p_α at second order in ϵ, which we denote by angle brackets. Making use of the relation

$$\mathrm{Re}\left[A\,\mathrm{e}^{\mathrm{i}\phi_0}\,\mathrm{Re}\left[B\,\mathrm{e}^{\mathrm{i}\phi_0}\right]\right] = \frac{1}{2}\mathrm{Re}\left[AB\,\mathrm{e}^{2\mathrm{i}\phi_0} + A^*B\right] \tag{24}$$

and discarding the oscillatory terms, we find

$$\langle\dot{\Lambda}\rangle = \frac{1}{2}\,\mathrm{Re}\left[fk\left(\frac{\partial R^*}{\partial q_\beta}\frac{\partial R}{\partial p_\beta} - \frac{\partial R^*}{\partial p_\beta}\frac{\partial R}{\partial q_\beta}\right) - \mathrm{i}k^2\frac{\partial}{\partial\Lambda}(f|R|^2)\right], \tag{25}$$

$$\langle\dot{q}_\alpha\rangle = -\frac{1}{2}\,\mathrm{Re}\left[\mathrm{i}f\left(\frac{\partial^2 R^*}{\partial p_\alpha\partial q_\beta}\frac{\partial R}{\partial p_\beta} - \frac{\partial^2 R^*}{\partial p_\alpha\partial p_\beta}\frac{\partial R}{\partial q_\beta}\right) + k\frac{\partial}{\partial\Lambda}\left(fR\frac{\partial R^*}{\partial p_\alpha}\right)\right], \tag{26}$$

$$\langle\dot{p}_\alpha\rangle = \frac{1}{2}\,\mathrm{Re}\left[\mathrm{i}f\left(\frac{\partial^2 R^*}{\partial q_\alpha\partial q_\beta}\frac{\partial R}{\partial p_\beta} - \frac{\partial^2 R^*}{\partial q_\alpha\partial p_\beta}\frac{\partial R}{\partial q_\beta}\right) + k\frac{\partial}{\partial\Lambda}\left(fR\frac{\partial R^*}{\partial q_\alpha}\right)\right]. \tag{27}$$

So far we have not used the fact that f is a real quantity; the reason for this will become apparent later. Away from the resonance, f is real and finite and equations (25)–(27) become

$$\langle\dot{\Lambda}\rangle = \frac{\partial\mathcal{H}}{\partial\lambda} = 0, \tag{28}$$

$$\langle\dot{q}_\alpha\rangle = \frac{\partial\mathcal{H}}{\partial p_\alpha}, \tag{29}$$

$$\langle\dot{p}_\alpha\rangle = -\frac{\partial\mathcal{H}}{\partial q_\alpha}, \tag{30}$$

where

$$\mathcal{H} = \frac{1}{4}\left[-\mathrm{i}f\left(\frac{\partial R^*}{\partial q_\beta}\frac{\partial R}{\partial p_\beta} - \frac{\partial R^*}{\partial p_\beta}\frac{\partial R}{\partial q_\beta}\right) - k\frac{\partial}{\partial\Lambda}(f|R|^2)\right] \tag{31}$$

is a real Hamiltonian.

The implication of our analysis is that each oscillatory term $R_{k,k'}$ in the disturbing function gives rise to an oscillatory behaviour in the disc at first order and to a secular behaviour at second order. Since it is described by a Hamiltonian $\mathcal{H}_{k,k'}$ that is independent of λ, this secular behaviour concerns only the eccentricity and inclination variables and is reversible and precessional in character. It is of interest mainly because it diverges in the vicinity of mean-motion resonances where $f \to \infty$. The resolution of this singularity, which results in non-Hamiltonian, irreversible secular behaviour, is discussed in Section 2.5.

The above procedure does not work for the secular part $R_{0,0}$ of the disturbing function, for which $\dot{\phi}_0$ vanishes identically. However, this term contributes directly to the secular dynamics at first order in the perturbation, giving rise to the familiar precessional effects, and we are not concerned with its smaller, second-order counterpart. Second-order effects are of interest only because of their amplification by mean-motion resonances. The secular dynamics is therefore adequately described by a Hamiltonian consisting of $-R_{0,0}$ together with the sum of $\mathcal{H}_{k,k'}$ for any important resonances $\{k,k'\}$, subject to the resolution of the corresponding singularities.

2.3 Complex canonical variables

Although Hamilton's equations employ real variables, complex variables are natural in celestial mechanics, especially for describing precessional behaviour. The idea of complex canonical variables was employed by Strocchi (1966) in order to unify the descriptions of classical and quantum-mechanical systems. Suppose we have a Hamiltonian system with canonical variables q_α and p_α and Hamiltonian $H(q_\alpha,p_\alpha)$.[5] Then (q_α,p_α) can be replaced by (z_α,z_α^*), where

$$z_\alpha=\frac{1}{\sqrt{2}}(q_\alpha+\mathrm{i}p_\alpha). \tag{32}$$

(In order for the dimensions to match, q_α and p_α should be variables of 'rectangular' rather than 'action–angle' form.) Hamilton's equations then combine in the compact form

$$\mathrm{i}\dot{z}_\alpha=\frac{\partial H}{\partial z_\alpha^*}, \tag{33}$$

in which H is now regarded as a real-valued analytic function of z_α and z_α^*, treated as algebraically independent variables. (The conjugate equation $-\mathrm{i}\dot{z}_\alpha^*=\partial H/\partial z_\alpha$ need not be considered separately, as it follows directly from the complex conjugate of equation (33); z_α and z_α^* are 'conjugate' variables in both senses of the word.)

The equations derived so far in this paper are valid for any choice of canonical eccentricity and inclination variables q_α and p_α used in conjunction with λ and Λ, such as modified Delaunay variables (e.g. Morbidelli 2002). Indeed the quantity

$$\frac{\partial R^*}{\partial q_\beta}\frac{\partial R}{\partial p_\beta}-\frac{\partial R^*}{\partial p_\beta}\frac{\partial R}{\partial q_\beta}=\{R^*,R\} \tag{34}$$

that appears in equation (31) is a Poisson bracket and is therefore invariant under canonical transformations. (The variables λ and Λ do not appear in this Poisson bracket because R is independent of λ.) We will use the canonical variables

$$q_1=\left\{2\Lambda\left[1-(1-e^2)^{1/2}\right]\right\}^{1/2}\cos\varpi, \tag{35}$$

$$p_1=\left\{2\Lambda\left[1-(1-e^2)^{1/2}\right]\right\}^{1/2}\sin\varpi, \tag{36}$$

$$q_2=\left[2\Lambda(1-e^2)^{1/2}(1-\cos i)\right]^{1/2}\cos\Omega, \tag{37}$$

$$p_2=\left[2\Lambda(1-e^2)^{1/2}(1-\cos i)\right]^{1/2}\sin\Omega, \tag{38}$$

which are identical to the rectangular variables of Poincaré (1892), but rotated through an angle of $\pi/2$. When $e\ll 1$ and $i\ll 1$, these variables reduce to $(q_1,p_1)\approx\Lambda^{1/2}(e\cos\varpi,e\sin\varpi)$ and $(q_2,p_2)\approx\Lambda^{1/2}(i\cos\Omega,i\sin\Omega)$, showing their relation to the Cartesian components of the eccentricity and inclination vectors (e.g. Murray & Dermott 1999).

Strocchi's transformation leads us to consider

$$z_1=\left\{\Lambda\left[1-(1-e^2)^{1/2}\right]\right\}^{1/2}\mathrm{e}^{\mathrm{i}\varpi}, \tag{39}$$

$$z_2=\left[\Lambda(1-e^2)^{1/2}(1-\cos i)\right]^{1/2}\mathrm{e}^{\mathrm{i}\Omega}, \tag{40}$$

which may be called *complex canonical Poincaré variables* and were also employed by Laskar & Robutel (1995).[6] Note that

$$L=\Lambda-|z_1|^2=[GMa(1-e^2)]^{1/2} \tag{41}$$

is the total specific angular momentum and

$$L_z=\Lambda-|z_1|^2-|z_2|^2=[GMa(1-e^2)]^{1/2}\cos i \tag{42}$$

is its component perpendicular to the reference plane. Since Λ is the specific angular momentum of a circular orbit of radius a in the reference plane, $|z_1|^2+|z_2|^2$ is the specific *angular momentum deficit* or AMD (Laskar 1997). It consists of two parts, $|z_1|^2$ associated with eccentricity and $|z_2|^2$ with inclination, and is a positive-definite measure of the deviation of an orbit from circularity and coplanarity (with respect to the arbitrary reference plane).

We now consider the complex potential amplitude R as an analytic function of Λ, z_α and z_α^* satisfying the analytic properties

[5] In this paragraph we assume that α labels all the degrees of freedom, not just a subset.

[6] They can alternatively be obtained from the modified Delaunay variables via the transformation $z_\alpha=p_\alpha^{1/2}\,\mathrm{e}^{-\mathrm{i}q_\alpha}$.

$$\left(\frac{\partial R}{\partial z_\alpha}\right)^* = \frac{\partial R^*}{\partial z_\alpha^*}, \qquad \left(\frac{\partial R}{\partial z_\alpha^*}\right)^* = \frac{\partial R^*}{\partial z_\alpha}. \tag{43}$$

The Poisson bracket of equation (34) becomes

$$\{R^*, R\} = -\mathrm{i}\left(\frac{\partial R^*}{\partial z_\beta}\frac{\partial R}{\partial z_\beta^*} - \frac{\partial R^*}{\partial z_\beta^*}\frac{\partial R}{\partial z_\beta}\right) \tag{44}$$

and so equation (31) for the second-order Hamiltonian reads

$$\mathcal{H} = \frac{1}{4}\left[f\left(\left|\frac{\partial R}{\partial z_\beta}\right|^2 - \left|\frac{\partial R}{\partial z_\beta^*}\right|^2\right) - k\frac{\partial}{\partial\Lambda}(f|R|^2)\right]. \tag{45}$$

It will be seen below that the terms in the first bracket, which arise from the dynamics of the eccentricity and inclination variables z_α, correspond in some sense to Lindblad and vertical resonances, while the term involving a derivative with respect to Λ, which arises from the dynamics of the mean-motion variables Λ and λ, corresponds in some sense to corotation resonances. A similar distinction is apparent in the analysis of Borderies, Goldreich & Tremaine (1984). We therefore refer to the first and second parts as the *Lindblad/vertical term* and the *corotation term*, respectively.

2.4 The disturbing function in complex canonical Poincaré variables

The disturbing function is usually expanded in the variables e and $s = \sin(i/2)$. In converting these into complex Poincaré variables it is useful to define the dimensionless complex eccentricity and inclination variables

$$\mathcal{E} = \left(\frac{2}{\Lambda}\right)^{1/2} z_1, \tag{46}$$

$$\mathcal{I} = \left(\frac{2}{\Lambda}\right)^{1/2} z_2, \tag{47}$$

and then to note that

$$e\,\mathrm{e}^{\mathrm{i}\varpi} = \mathcal{E}\left(1 - \tfrac{1}{4}|\mathcal{E}|^2\right)^{1/2}, \tag{48}$$

$$2s\,\mathrm{e}^{\mathrm{i}\Omega} = \mathcal{I}\left(1 - \tfrac{1}{2}|\mathcal{E}|^2\right)^{-1/2}. \tag{49}$$

Therefore the magnitudes of $\mathcal{E}$ and $\mathcal{I}$ are directly related to the eccentricity and inclination and are approximately equal to them when small, while the phases of $\mathcal{E}$ and $\mathcal{I}$ are the corresponding precessional longitudes ϖ and Ω.

Expansions for R in terms of $\mathcal{E}$ and $\mathcal{I}$ up to fourth degree in eccentricity and inclination are given for the most important commensurabilities in Appendix A. Laskar & Robutel (1995) provide a method for generating such expansions, and prove theorems regarding their general algebraic form. In fact, the expressions in Appendix A were obtained simply by rewriting the expansion of Murray & Dermott (1999) in terms of $\mathcal{E}$ and $\mathcal{I}$. (Note, however, that primed variables always refer here to the companion, regardless of whether the resonance lies interior or exterior to it.)

For the purposes of illustration, consider a first-order interior $j : j-1$ resonance with $j \geq 2$. By this notation we mean that the resonant orbit in the disc is interior to the perturber and has a mean motion $j/(j-1)$ times that of the companion. We define the ratio of semimajor axes $\alpha = a/a' = [(j-1)/j]^{2/3}$. Murray & Dermott (1999) give the disturbing function as the sum of direct and indirect terms in the form

$$\mathcal{R} = \frac{GM'}{a'}\left(\mathcal{R}_\mathrm{D} + \alpha\mathcal{R}_\mathrm{E}\right). \tag{50}$$

The terms of lowest degree in eccentricity and inclination for this commensurability are (Murray & Dermott 1999, Tables B.4 and B.5)

$$\mathcal{R}_\mathrm{D} = ef_{27}\cos[j\lambda' - (j-1)\lambda - \varpi] + e'f_{31}\cos[j\lambda' - (j-1)\lambda - \varpi'], \tag{51}$$

$$\mathcal{R}_\mathrm{E} = -2e'\cos[j\lambda' - (j-1)\lambda - \varpi']\delta_{j,2}, \tag{52}$$

where δ is the Kronecker delta and

$$f_{27} = \frac{1}{2}\left(-2j - \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)b_{1/2}^{(j)}(\alpha), \tag{53}$$

$$f_{31} = \frac{1}{2}\left(-1 + 2j + \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)b_{1/2}^{(j-1)}(\alpha) \tag{54}$$

are given in terms of Laplace coefficients. Comparing with equation (4), we may therefore identify the integer coefficients $k = j-1$ and $k' = -j$, the corresponding frequency $\dot{\phi}_0 = (j-1)n - jn'$ and the complex potential amplitude

$$R \approx \frac{GM'}{a'}\left[f_{27}\mathcal{E} + (f_{31} - 2\alpha\delta_{j,2})\mathcal{E}'\right]. \tag{55}$$

The equivalent expression correct to fourth degree in eccentricity and inclination is given in equation (A3), which compactly combines all terms in Tables B.4 and B.5 of Murray & Dermott (1999). Note that the inclination terms vanish when $\mathcal{I} = \mathcal{I}'$, i.e. when the disc and perturber are coplanar.

2.5 Resolution of the resonant singularity

As a simple but important example, at a second-order interior resonance with a perturber having a circular and coplanar orbit, such as the 3 : 1 resonance that occurs in accretion discs in close binary stars of sufficiently small mass ratio, the dominant term in the disturbing function has $R \propto \mathcal{E}^2$ for $e \ll 1$ and is therefore of the form

$$R = g(\Lambda)z_1^2 + O(z^4), \tag{56}$$

where, in fact, g is real (cf. equation A5). According to equation (45), the second-order Hamiltonian arising from this disturbing function is

$$\mathcal{H} = \dot{\phi}_0^{-1} g^2 |z_1|^2 + O(z^4), \tag{57}$$

which gives rise to the dynamics

$$\mathrm{i}\langle\dot{z}_1\rangle = \frac{\partial\mathcal{H}}{\partial z_1^*} = \dot{\phi}_0^{-1} g^2 z_1 + O(z^3). \tag{58}$$

Since $\arg(z_1) = \varpi$, this behaviour corresponds simply to apsidal precession at a rate $-g^2/\dot{\phi}_0$, which is retrograde interior to the resonance (where $\dot{\phi}_0 > 0$), prograde exterior to the resonance (where $\dot{\phi}_0 < 0$) and diverges at the resonance itself. This behaviour can be seen, for example, in Fig. 1 of Knežević et al. (1991), which shows how the mean apsidal and nodal precession rates of asteroids in nearly circular orbits, calculated to second order in the planetary masses, are affected in the vicinity of mean-motion resonances with Jupiter. The 3 : 1 and 5 : 3 resonances display the characteristic behaviour described here for second-order interior resonances.

This divergence arises because the expansion we have adopted breaks down in the vicinity of the resonance. The behaviour of non-interacting particles and of continuous discs near a mean-motion resonance is different. Single particles avoid the divergence by undergoing libration, which means that the resonant angle, instead of circulating arbitrarily slowly, oscillates about an equilibrium value. Fluid or collisional particle discs, in which the intersection of streamlines is resisted, can avoid the divergence through the intervention of collective effects such as pressure, self-gravity or viscosity. Meyer-Vernet & Sicardy (1987) showed how any of these collective effects, in the case of a Lindblad resonance, acts to displace the resonant pole slightly away from the real axis, after a spatial Fourier transform of the linearized perturbation equations is carried out. We initially adopt a similar prescription, replacing equation (18) with

$$f = \frac{1}{\dot{\phi}_0 - \mathrm{i}s}, \tag{59}$$

where s is a positive parameter with the dimensions of frequency. This replacement can also be motivated by Landau's approach, in which the periodic disturbing function is 'turned on' slowly by including an additional time-dependence proportional to e^{st} (cf. Borderies, Goldreich & Tremaine 1984). However, Landau's approach must be used with care and we return to address this issue more thoroughly in Section 2.6.

When f is treated throughout the analysis of Section 2.2 as a complex quantity, we obtain secular evolutionary equations at second order that are not of Hamiltonian form but allow for the irreversibility that is associated with the resonance. In particular,

$$\langle\dot{\Lambda}\rangle = -\frac{1}{2}kf_\mathrm{i}\left(\left|\frac{\partial R}{\partial z_\beta}\right|^2 - \left|\frac{\partial R}{\partial z_\beta^*}\right|^2\right) + \frac{1}{2}k^2\frac{\partial}{\partial\Lambda}(f_\mathrm{i}|R|^2), \tag{60}$$

$$\mathrm{i}\langle\dot{z}_\alpha\rangle = \frac{1}{4}f\left(\frac{\partial^2 R^*}{\partial z_\alpha^*\partial z_\beta^*}\frac{\partial R}{\partial z_\beta} - \frac{\partial^2 R^*}{\partial z_\alpha^*\partial z_\beta}\frac{\partial R}{\partial z_\beta^*}\right) + \frac{1}{4}f^*\left(\frac{\partial^2 R}{\partial z_\alpha^*\partial z_\beta}\frac{\partial R^*}{\partial z_\beta^*} - \frac{\partial^2 R}{\partial z_\alpha^*\partial z_\beta^*}\frac{\partial R^*}{\partial z_\beta}\right) - \frac{1}{4}k\frac{\partial}{\partial\Lambda}\left(fR\frac{\partial R^*}{\partial z_\alpha^*} + f^*R^*\frac{\partial R}{\partial z_\alpha^*}\right), \tag{61}$$

where $f_\mathrm{i} = \mathrm{Im}[f]$; these expressions agree with $\partial\mathcal{H}/\partial\lambda$ and $\partial\mathcal{H}/\partial z_\alpha^*$ if $f_\mathrm{i} = 0$. In the present example we obtain, for $e \ll 1$,

$$\mathrm{i}\langle\dot{z}_1\rangle = fg^2 z_1 = \left(\frac{\dot{\phi}_0 + \mathrm{i}s}{\dot{\phi}_0^2 + s^2}\right) g^2 z_1, \tag{62}$$

instead of equation (58). Resolution of the resonant singularity therefore leads to a finite precession rate of $-g^2\dot{\phi}_0/(\dot{\phi}_0^2 + s^2)$ together with a qualitatively different effect: an eccentricity growth rate of $g^2 s/(\dot{\phi}_0^2 + s^2)$. The growth rate peaks at the resonance, the height and width of the peak depending on the parameter s, while the integrated growth rate is independent of s. In fact the growth rate can be represented as $\pi g^2\delta(\dot\phi_0)$ as $s\to 0$, i.e. in the case of a resonance that is not significantly broadened. We note also that

$$\delta(\dot\phi_0)=\left|\frac{\mathrm{d}\dot\phi_0}{\mathrm{d}\Lambda_0}\right|^{-1}\delta(\Lambda_0-\hat\Lambda_0)=\left|\frac{\mathrm{d}\dot\phi_0}{\mathrm{d}a_0}\right|^{-1}\delta(a_0-\hat a_0), \tag{63}$$

where $\hat\Lambda_0$ (or $\hat a_0$) locates the resonant orbit. Unlike the Hamiltonian precessional behaviour, the eccentricity growth is irreversible and corresponds to a growth of the AMD of the disc:

$$\left\langle\frac{\mathrm{d}|z_1|^2}{\mathrm{d}t}\right\rangle=\left(\frac{2s}{\dot\phi_0^2+s^2}\right)g^2|z_1|^2\sim 2\pi g^2|z_1|^2\,\delta(\dot\phi_0). \tag{64}$$

The physical resolution of the divergent precession rate is therefore a finite precession rate together with an irreversible growth or decay (in this case a growth of eccentricity). This behaviour is characteristic of an eccentric Lindblad resonance as described by Lubow (1991). For the $3:1$ resonance in a system with mass ratio $q=M'/M\ll 1$ we obtain $g\approx 1.727\,q(GM/a'^3)^{1/2}$ and an eccentricity growth rate of $2.082\,q^2(GM/a'^3)^{1/2}a_0\,\delta(a_0-\hat a_0)$. This agrees with the result of Lubow (1991) as interpreted by Goodchild & Ogilvie (2006). However, neither of these papers considered the second-order precessional effect.

Applying equation (14) of Borderies, Goldreich & Tremaine (1984), however, gives exactly *twice* the eccentricity growth rate of Lubow (1991). The reason for this is that their (modified Delaunay) variable Γ is proportional to e^2 for $e\ll 1$. In their analysis e^2 grows partly because the non-oscillatory part of e grows secularly, which is physically correct, but also because the oscillatory part of e grows proportionally to e^{st}. In reality the oscillatory part of e has a steady amplitude owing to dissipation in the disc. Therefore application of Landau's prescription in this case gives a misleading result for the second-order behaviour of the disc.

2.6 Collective effects in the disc

We now attempt to give a justification for the procedure described above. The detailed behaviour of the disc near a resonance depends, in principle, on a number of factors: the relative importance of various collective effects (such as pressure, self-gravity or viscosity), the level of nonlinearity, the vertical structure of a three-dimensional disc, etc. It is known that the torque exerted at a Lindblad resonance is independent of these details provided that the response is linear and localized near the resonant orbit (Meyer-Vernet & Sicardy 1987; Lubow & Ogilvie 1998), while nonlinearity in fact makes little difference (Yuan & Cassen 1994). On the other hand, the torque exerted at a corotation resonance does depend in an important way on the relative importance of nonlinearity and viscosity (Ogilvie & Lubow 2003).

We therefore adopt a minimal description of the collective effects in the disc, by introducing viscous behaviour without attempting the formidable task of expressing the full Navier–Stokes equation in terms of orbital elements. The dynamical equations adopted so far refer to individual particles, and the orbital elements of a given particle are functions of time only. To deal with a continuous disc we adopt a semi-Lagrangian description with spatial coordinates (a_0,λ_0) labelling the fluid elements at any instant.[7] Note that the unperturbed variables satisfy $a_0=$ constant and $\lambda_0-n_0t=\epsilon_0=$ constant for any fluid element, with $n_0=(GM/a_0^3)^{1/2}$. Therefore (a_0,ϵ_0) are true Lagrangian coordinates but are less suitable because of the rapid shearing in the unperturbed flow. The total time-derivative $\mathrm{d}/\mathrm{d}t$ translates into the Lagrangian derivative

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+n_0\frac{\partial}{\partial\lambda_0}. \tag{65}$$

Even a simplified model of collective effects has certain minimal requirements. The terms added to the dynamical equations should regularize the resonant singularity that occurs in the first-order linearized response. At the same time they should have the correct form to conserve angular momentum while dissipating energy. Since viscosity is known to be required to resolve the singularity in the linearized equations at a corotation resonance, we add diffusive terms to each equation for the purposes of regularization. The diffusion coefficient ν can be identified as the effective viscosity of the disc. Specifically, we adopt the model

$$\frac{\mathrm{D}\lambda}{\mathrm{D}t}=\frac{\partial H}{\partial\Lambda}+\left(\frac{\mathrm{d}m}{\mathrm{d}a_0}\right)^{-1}\frac{\partial}{\partial a_0}\left(\nu\frac{\mathrm{d}m}{\mathrm{d}a_0}\frac{\partial\lambda}{\partial a_0}\right), \tag{66}$$

$$\frac{\mathrm{D}\Lambda}{\mathrm{D}t}=-\frac{\partial H}{\partial\lambda}+\left(\frac{\mathrm{d}m}{\mathrm{d}a_0}\right)^{-1}\frac{\partial}{\partial a_0}\left(\nu\frac{\mathrm{d}m}{\mathrm{d}a_0}\frac{\partial\Lambda}{\partial a_0}\right)-2\nu\left|\frac{\partial z_\beta}{\partial a_0}\right|^2, \tag{67}$$

$$\frac{\mathrm{D}z_\alpha}{\mathrm{D}t}=-\mathrm{i}\frac{\partial H}{\partial z_\alpha^*}+\left(\frac{\mathrm{d}m}{\mathrm{d}a_0}\right)^{-1}\frac{\partial}{\partial a_0}\left(\nu\frac{\mathrm{d}m}{\mathrm{d}a_0}\frac{\partial z_\alpha}{\partial a_0}\right), \tag{68}$$

[7] For a three-dimensional description a third coordinate is required to describe the distance from the 'midplane' of the disc.

where $m(a_0)$ is the cumulative mass function of the disc, i.e. the total mass interior to the orbit labelled by a_0. (For a circular disc, $\mathrm{d}m/\mathrm{d}a_0 = 2\pi\Sigma a_0$, where Σ is the surface density.) The reason for writing the diffusive terms in this way, rather than simply as $\nu(\partial^2\Lambda/\partial a_0^2)$, etc., is so that they properly conserve the mass-weighted integrated quantities. Now if a certain quantity obeys a diffusion equation, the square of that quantity is both diffused and dissipated.[8] By allowing z_α to diffuse we therefore allow the specific AMD $|z_\beta|^2$ to diffuse and dissipate. The same dissipative term is added to the 'energy' equation (67) so that the specific angular momentum $L_z = \Lambda - |z_\beta|^2$ diffuses conservatively and does not dissipate.

To see how this model operates, consider the behaviour in the absence of a perturber. With suitable boundary conditions we have, after an integration by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |z_\beta|^2\,\mathrm{d}m = -2\int \nu\left|\frac{\partial z_\beta}{\partial a_0}\right|^2\mathrm{d}m, \tag{69}$$

showing the dissipation of the total AMD.[9] Energy (or Λ) is also dissipated, but the total angular momentum perpendicular to the reference plane is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\Lambda - |z_\beta|^2\right)\,\mathrm{d}m = 0. \tag{70}$$

In these integrals the mass element is

$$\mathrm{d}m = \frac{1}{2\pi}\frac{\mathrm{d}m}{\mathrm{d}a_0}\,\mathrm{d}a_0\,\mathrm{d}\lambda_0. \tag{71}$$

We may neglect the slow effects of diffusion on the smooth unperturbed state. Consider now a first-order equation such as

$$\frac{\mathrm{D}\Lambda_1}{\mathrm{D}t} = \mathrm{Re}\left[\mathrm{i}kR\,\mathrm{e}^{\mathrm{i}\phi_0}\right] + \left(\frac{\mathrm{d}m}{\mathrm{d}a_0}\right)^{-1}\frac{\partial}{\partial a_0}\left(\nu\frac{\mathrm{d}m}{\mathrm{d}a_0}\frac{\partial\Lambda_1}{\partial a_0}\right). \tag{72}$$

Unlike the ordinary differential equation (12), this is now a partial differential equation in variables (a_0, λ_0, t). Since the equation is linear and the forcing term depends on λ_0 and t only through the exponential, the solution is of the form $\Lambda_1 = \mathrm{Re}\left[\tilde\Lambda_1(a_0)\,\mathrm{e}^{\mathrm{i}\phi_0}\right]$, where $\tilde\Lambda_1$ satisfies the ordinary differential equation

$$\mathrm{i}\dot\phi_0\tilde\Lambda_1 = \mathrm{i}kR + \left(\frac{\mathrm{d}m}{\mathrm{d}a_0}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}a_0}\left(\nu\frac{\mathrm{d}m}{\mathrm{d}a_0}\frac{\mathrm{d}\tilde\Lambda_1}{\mathrm{d}a_0}\right). \tag{73}$$

It is clear that the viscous term is required to regularize the solution at the resonance where $\dot\phi_0 = 0$. For a localized response the variation of all coefficients with a_0 can be neglected, except for $\dot\phi_0 \approx Dx$, where $x = a_0 - \hat a_0$ is the distance from the resonance and $D = \mathrm{d}\dot\phi_0/\mathrm{d}a_0$ is the rate of detuning. Then $\tilde\Lambda_1 = fkR$, where $f(x)$ now denotes the solution (decaying as $x \to \pm\infty$) of the rescaled equation with unit forcing,

$$Dxf + \mathrm{i}\nu\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 1. \tag{74}$$

Standard Fourier-transform methods (cf. Meyer-Vernet & Sicardy 1987) give

$$f(x) = \frac{\mathrm{i}}{|D|}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx+(\nu/3D)k^3}H(-k\,\mathrm{sgn}D)\,\mathrm{d}k, \tag{75}$$

where H is the Heaviside step function. Note that $f \approx (Dx)^{-1} \approx \dot\phi_0^{-1}$ far from the resonance. In fact f resembles $(Dx - \mathrm{i}s)^{-1}$ for all x (Fig. 1) if $s \approx (\nu D^2)^{1/3}$. Furthermore

$$\int_{-\infty}^{\infty} f_\mathrm{i}\,\mathrm{d}x = \int_{-\infty}^{\infty}\nu\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|^2\mathrm{d}x = \frac{\pi}{|D|}. \tag{76}$$

Therefore the resolution of the resonant singularity by viscosity gives a solution with properties very similar to that of the simple replacement (59). In particular, the integrated effect of the imaginary part of f, which gives rise to the irreversible behaviour, is independent of ν and identical to that obtained using equation (59).

The solution for λ_1 and $z_{\alpha 1}$ follows in the same way, with the complex $f(x)$ replacing the $\dot\phi_0^{-1}$ of the original analysis of Section 2.2.[10] The solution proceeds similarly to the second order, with one important exception. The dissipative term in the

[8] Specifically, if $\partial_t u = \partial_{xx} u$ and $v = |u|^2$, then $\partial_t v = \partial_{xx} v - 2|\partial_x u|^2$.

[9] A slightly more sophisticated model would allow the dissipation to be related to gradients of z_α multiplied by some function of a_0, e.g. if the gradient of $\mathcal{I}$, rather than that of z_2, is the important quantity in a warped disc. However, such refinements are inconsequential in this context.

[10] It is easy to generalize this treatment of collective effects to allow different viscosities to act on the mean-motion variables (Λ, λ), the eccentricity variable z_1 and the inclination variable z_2. Complex viscosities are also permissible, giving solutions in the form of attenuated waves emitted from the resonance.

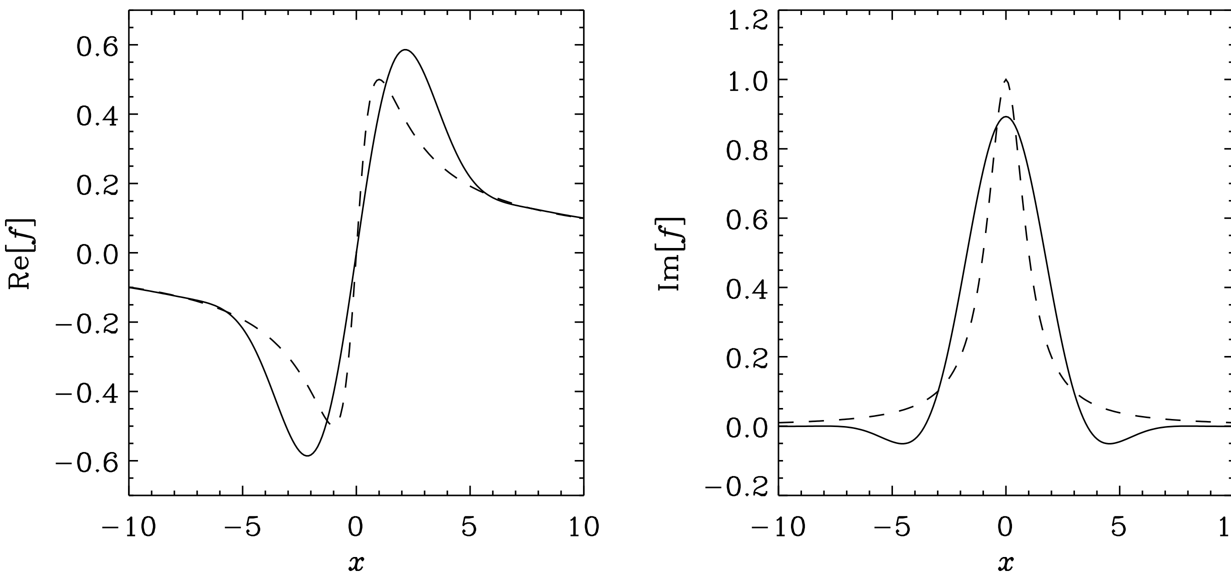

Figure 1. Real and imaginary parts of the complex function $f(x)$, which represents how the resonant singularity at $x = 0$ is resolved. The solid line shows the viscous solution, equation (75), while the dashed line gives the simpler 'damped' model $f = (Dx - \mathrm{i}s)^{-1}$ for $s = 1$. Without loss of generality, units are chosen such that $D = \nu = 1$.

'energy' equation (67) provides an additional contribution to $\langle\dot{\Lambda}\rangle$, which is needed to enforce angular momentum conservation. Using equation (76) and carrying out an integration by parts, we find

$$\int \langle\dot{\Lambda}\rangle_{\mathrm{d}}\,\mathrm{d}m = -\frac{\pi}{2|D|}\left[(k+1)\left|\frac{\partial R}{\partial z_\beta}\right|^2\frac{\mathrm{d}m}{\mathrm{d}a_0} - (k-1)\left|\frac{\partial R}{\partial z_\beta^*}\right|^2\frac{\mathrm{d}m}{\mathrm{d}a_0} + k^2|R|^2\frac{\mathrm{d}}{\mathrm{d}a_0}\left(\frac{\mathrm{d}m}{\mathrm{d}\Lambda_0}\right)\right]. \tag{77}$$

Here $\langle\dot{\Lambda}\rangle_{\mathrm{d}}$ includes the dissipative contribution, which has the effect of changing k into $k \pm 1$. If the disc remains circular, Λ is the specific angular momentum and the above rate of change of action corresponds to the torque on the disc. It agrees in form with the torque formulae for Lindblad and corotation resonances (Goldreich & Tremaine 1979). Note that $\mathrm{d}m/\mathrm{d}\Lambda_0$ ($= 4\pi\Sigma/n$) is inversely proportional to the vortensity of a circular Keplerian disc, so the corotation torque is proportional to the gradient of inverse vortensity while the Lindblad/vertical torques are proportional to the surface density.

It appears that this procedure cannot be followed if action–angle eccentricity and inclination variables are used as in Goldreich & Tremaine (1981) and Borderies, Goldreich & Tremaine (1984). We have found it essential to allow eccentricity and inclination to diffuse (or propagate) at first order, to resolve the resonant singularity, and for the accompanying dissipation to appear at second order. This is most naturally achieved by using complex variables for eccentricity and inclination, to which the energy and AMD are related quadratically.

3 THE UNRESTRICTED PROBLEM

3.1 Canonical equations

We now extend the problem to treat dynamically the orbit of the companion of mass M' around the central mass M. We adopt canonical variables $(\lambda', \Lambda', q'_\alpha, p'_\alpha)$ for the two-body problem, where $\Lambda' = [G(M+M')a']^{1/2}$ and q'_α and p'_α are defined by analogy with equations (35)–(38).

The Hamiltonian of the unperturbed two-body problem is $\mu H'_0$, where $\mu = MM'/(M+M')$ is the reduced mass and

$$H'_0 = -\frac{[G(M+M')]^2}{2\Lambda'^2} = -\frac{G(M+M')}{2a'}, \tag{78}$$

giving rise to an unperturbed mean motion

$$\dot{\lambda}' = n'(\Lambda') = \frac{\mathrm{d}H'_0}{\mathrm{d}\Lambda'} = \left[\frac{G(M+M')}{a'^3}\right]^{1/2}. \tag{79}$$

The total Hamiltonian of the coupled system is then of the form

$$H = \int \left\{ H_0(\Lambda) - \epsilon \,\mathrm{Re} \left[R(\Lambda, q_\alpha, p_\alpha, \Lambda', q'_\alpha, p'_\alpha)\, \mathrm{e}^{\mathrm{i}\phi} \right] \right\} \,\mathrm{d}m + \mu H'_0(\Lambda'), \tag{80}$$

and Hamilton's equations read

$$\dot{\lambda} = n(\Lambda) - \epsilon \,\mathrm{Re} \left[\frac{\partial R}{\partial \Lambda}\, \mathrm{e}^{\mathrm{i}\phi} \right], \tag{81}$$

$$\dot{\Lambda} = \epsilon \,\mathrm{Re} \left[\mathrm{i}kR\, \mathrm{e}^{\mathrm{i}\phi} \right], \tag{82}$$

$$\dot{q}_\alpha = -\epsilon \,\mathrm{Re} \left[\frac{\partial R}{\partial p_\alpha}\, \mathrm{e}^{\mathrm{i}\phi} \right], \tag{83}$$

$$\dot{p}_\alpha = \epsilon \,\mathrm{Re} \left[\frac{\partial R}{\partial q_\alpha}\, \mathrm{e}^{\mathrm{i}\phi} \right], \tag{84}$$

as for the restricted problem, together with

$$\dot{\lambda}' = n'(\Lambda') - \frac{\epsilon}{\mu} \int \mathrm{Re} \left[\frac{\partial R}{\partial \Lambda'}\, \mathrm{e}^{\mathrm{i}\phi} \right] \,\mathrm{d}m, \tag{85}$$

$$\dot{\Lambda}' = \frac{\epsilon}{\mu} \int \mathrm{Re} \left[\mathrm{i}k' R\, \mathrm{e}^{\mathrm{i}\phi} \right] \,\mathrm{d}m, \tag{86}$$

$$\dot{q}'_\alpha = -\frac{\epsilon}{\mu} \int \mathrm{Re} \left[\frac{\partial R}{\partial p'_\alpha}\, \mathrm{e}^{\mathrm{i}\phi} \right] \,\mathrm{d}m, \tag{87}$$

$$\dot{p}'_\alpha = \frac{\epsilon}{\mu} \int \mathrm{Re} \left[\frac{\partial R}{\partial q'_\alpha}\, \mathrm{e}^{\mathrm{i}\phi} \right] \,\mathrm{d}m, \tag{88}$$

determining the feedback on the two-body orbit.

3.2 Perturbative solution

The solution can be developed in powers of ϵ as in Section 2.2. When it comes to calculating the secular evolution at $O(\epsilon^2)$ there are no cross-contributions from different regions of the disc because their first-order oscillations are out of phase. We then find that the secular dynamics at second order, away from resonance, is governed by the Hamiltonian

$$\int \mathcal{H} \,\mathrm{d}m \tag{89}$$

where $\mathcal{H}$ is given by equation (31) or equation (45), as in the restricted problem, except that $\mathcal{H}$ is now a function of $(\Lambda, q_\alpha, p_\alpha, \Lambda', q'_\alpha, p'_\alpha)$. In detail,

$$\langle \dot{\Lambda} \rangle = \frac{\partial \mathcal{H}}{\partial \lambda} = 0, \tag{90}$$

$$\mathrm{i} \langle \dot{z}_\alpha \rangle = \frac{\partial \mathcal{H}}{\partial z^*_\alpha}, \tag{91}$$

$$\langle \dot{\Lambda}' \rangle = \frac{1}{\mu} \int \frac{\partial \mathcal{H}}{\partial \lambda'} \,\mathrm{d}m = 0, \tag{92}$$

$$\mathrm{i} \langle \dot{z}'_\alpha \rangle = \frac{1}{\mu} \int \frac{\partial \mathcal{H}}{\partial z^*_\alpha} \,\mathrm{d}m. \tag{93}$$

The total AMD of the system is conserved under this precessional dynamics because

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(\int |z_\beta|^2 \,\mathrm{d}m + \mu |z'_\beta|^2 \right) \right\rangle = \mathrm{i} \int \left(z_\beta \frac{\partial \mathcal{H}}{\partial z_\beta} - z^*_\beta \frac{\partial \mathcal{H}}{\partial z^*_\beta} + z'_\beta \frac{\partial \mathcal{H}}{\partial z'_\beta} - z'^*_\beta \frac{\partial \mathcal{H}}{\partial z'^*_\beta} \right) \mathrm{d}m = -\int \frac{\partial \mathcal{H}}{\partial \theta} \,\mathrm{d}m = 0, \tag{94}$$

where θ is an angle of rotation of the entire system about the z-axis. (Under a rotation through $\delta\theta$, ϖ and Ω decrease by $\delta\theta$ and so $\delta z_\alpha = -\mathrm{i} z_\alpha \,\delta\theta$.) This conservation law is in addition to the conservation of $\int \mathcal{H} \,\mathrm{d}m$ itself.

Resolution of the resonant singularity can again be achieved using the replacement (59) or a more sophisticated version thereof. We find

$$\langle \dot{\Lambda} \rangle = -\frac{1}{2} k f_\mathrm{i} \left(\left| \frac{\partial R}{\partial z_\beta} \right|^2 - \left| \frac{\partial R}{\partial z^*_\beta} \right|^2 \right) + \frac{1}{2} k^2 \frac{\partial}{\partial \Lambda} (f_\mathrm{i} |R|^2), \tag{95}$$

$$\mathrm{i}\langle\dot{z}_\alpha\rangle=\frac{1}{4}f\left(\frac{\partial^2R^*}{\partial z_\alpha^*\partial z_\beta^*}\frac{\partial R}{\partial z_\beta}-\frac{\partial^2R^*}{\partial z_\alpha^*\partial z_\beta}\frac{\partial R}{\partial z_\beta^*}\right)+\frac{1}{4}f^*\left(\frac{\partial^2R}{\partial z_\alpha^*\partial z_\beta}\frac{\partial R^*}{\partial z_\beta^*}-\frac{\partial^2R}{\partial z_\alpha^*\partial z_\beta^*}\frac{\partial R^*}{\partial z_\beta}\right)-\frac{1}{4}k\frac{\partial}{\partial\Lambda}\left(fR\frac{\partial R^*}{\partial z_\alpha^*}+f^*R^*\frac{\partial R}{\partial z_\alpha^*}\right),\tag{96}$$

$$\langle\dot{\Lambda}'\rangle=\frac{1}{\mu}\int\left[-\frac{1}{2}k'f_\mathrm{i}\left(\left|\frac{\partial R}{\partial z_\beta}\right|^2-\left|\frac{\partial R}{\partial z_\beta^*}\right|^2\right)+\frac{1}{2}kk'\frac{\partial}{\partial\Lambda}(f_\mathrm{i}|R|^2)\right]\mathrm{d}m,\tag{97}$$

$$\begin{aligned}\mathrm{i}\langle\dot{z}'_\alpha\rangle=\frac{1}{\mu}\int\Bigg[&\frac{1}{4}f\left(\frac{\partial^2R^*}{\partial z_\alpha'^*\partial z_\beta^*}\frac{\partial R}{\partial z_\beta}-\frac{\partial^2R^*}{\partial z_\alpha'^*\partial z_\beta}\frac{\partial R}{\partial z_\beta^*}\right)+\frac{1}{4}f^*\left(\frac{\partial^2R}{\partial z_\alpha'^*\partial z_\beta}\frac{\partial R^*}{\partial z_\beta^*}-\frac{\partial^2R}{\partial z_\alpha'^*\partial z_\beta^*}\frac{\partial R^*}{\partial z_\beta}\right)\\&-\frac{1}{4}k\frac{\partial}{\partial\Lambda}\left(fR\frac{\partial R^*}{\partial z_\alpha'^*}+f^*R^*\frac{\partial R}{\partial z_\alpha'^*}\right)\Bigg]\mathrm{d}m.\end{aligned}\tag{98}$$

Equation (95) requires modification to account for dissipation, as discussed in Section 2.6, leading again to equation (77).

According to this analysis the secular behaviour of the disc is governed by the precisely same equations as in the restricted problem, but the orbital elements of the companion now also evolve in a consistent way through the feedback of the disc on the two-body orbit. This coupled dynamics is of special interest when the disc and companion have comparable angular momenta, as may occur in protoplanetary systems or planetary rings.

4 APPLICATIONS

We now work out some of the implications of the preceding analysis. We restrict attention initially to effects of first degree in eccentricity and inclination, and then consider an example of nonlinear behaviour in Section 4.4.

4.1 Second-order resonances

In Section 2.5 we considered the behaviour of eccentricity at an interior second-order resonance in the circular restricted problem. We now generalize the result to permit both the disc and the companion to have small eccentricities and inclinations, and to allow dynamical feedback on the companion's orbit.

For an interior second-order $j:j-2$ resonance we have, to lowest degree in eccentricity and inclination,

$$\mathrm{i}\langle\dot{z}_1\rangle=\frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda_0}\right)^{3/2}f\left[2f_{45}(2f_{45}\mathcal{E}+f_{49}\mathcal{E}')\right],\tag{99}$$

$$\mathrm{i}\langle\dot{z}_2\rangle=\frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda_0}\right)^{3/2}f\left[\frac{1}{4}f_{57}^2(\mathcal{I}-\mathcal{I}')\right],\tag{100}$$

$$\mathrm{i}\langle\dot{z}'_1\rangle=\frac{1}{\mu}\int\frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda'_0}\right)^{1/2}\left(\frac{2}{\Lambda_0}\right)f\left[f_{49}(2f_{45}\mathcal{E}+f_{49}\mathcal{E}')\right]\mathrm{d}m,\tag{101}$$

$$\mathrm{i}\langle\dot{z}'_2\rangle=\frac{1}{\mu}\int\frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda'_0}\right)^{1/2}\left(\frac{2}{\Lambda_0}\right)f\left[\frac{1}{4}f_{57}^2(\mathcal{I}'-\mathcal{I})\right]\mathrm{d}m.\tag{102}$$

These expressions, which follow from the terms of second degree in the disturbing function (A5), derive only from the 'Lindblad/vertical' aspects of the resonance; there are no 'corotation' terms here. The rate of change of total AMD is

$$\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\left(\int|z_\beta|^2\,\mathrm{d}m+\mu|z'_\beta|^2\right)\right\rangle=\int\frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda_0}\right)2f_\mathrm{i}\left(|2f_{45}\mathcal{E}+f_{49}\mathcal{E}'|^2+\frac{1}{4}f_{57}^2|\mathcal{I}-\mathcal{I}'|^2\right)\mathrm{d}m.\tag{103}$$

For a localized resonance f_i is strongly peaked and we may use equation (76) to express the result as

$$\left(\frac{GM'}{a'}\right)^2\frac{\pi}{\Lambda_0|D|}\frac{\mathrm{d}m}{\mathrm{d}a_0}\left[|2f_{45}\mathcal{E}+f_{49}\mathcal{E}'|^2+\frac{1}{4}f_{57}^2|\mathcal{I}-\mathcal{I}'|^2\right],\tag{104}$$

evaluated at the resonance. The equivalent result for an exterior second-order resonance is

$$\left(\frac{GM'}{a}\right)^2\frac{\pi}{\Lambda_0|D|}\frac{\mathrm{d}m}{\mathrm{d}a_0}\left[|2(f_{53}-\tfrac{3}{8}\alpha^{-2}\delta_{j,3})\mathcal{E}+f_{49}\mathcal{E}'|^2+\frac{1}{4}f_{57}^2|\mathcal{I}-\mathcal{I}'|^2\right].\tag{105}$$

The meaning of these results is that second-order resonances allow an exponential growth of AMD for small eccentricities and inclinations. We have already explained the connection with the result of Lubow (1991) concerning the growth of eccentricity in the disc when the companion has a fixed circular orbit. (It is noteworthy that the indirect term affects, and largely cancels, the eccentricity growth rate associated with the 1 : 3 resonance in an exterior disc, so it should always be included in studies of satellite–disc interactions.) On the other hand, if the disc is fixed artificially to be circular, the eccentricity of the companion can grow exponentially. Using the fact that $f_{49} \approx -(j^2/2\pi)[5K_0(4/3) + (19/4)K_1(4/3)]$ for $j \gg 1$, where K is the modified Bessel function, we find a growth rate for $\mathcal{E}'$ that agrees with equation (8) of the analysis of eccentric Lindblad resonances by Ward (1988)[11] in the case of a circular disc and $e' \ll 1$.

For the first time, though, we see here the effect of the resonance in the general case when both the disc and companion are eccentric. Certain linear combinations of the complex variables $\mathcal{E}$ and $\mathcal{E}'$ are involved, which depend on j. The case of inclination is easier to understand, as the relevant combination is always $\mathcal{I} - \mathcal{I}'$, the complex 'mutual inclination' of the disc and companion. Indeed, Lubow & Ogilvie (2001) were able to write down such equations for inclination dynamics based on the formulae of Borderies, Goldreich & Tremaine (1984) (for a fixed disc) and the idea that only the complex mutual inclination could enter. Using the fact that $f_{57} = (\alpha/2)b_{3/2}^{(j-1)}(\alpha)$, we find agreement with equation (48) of Lubow & Ogilvie (2001). Although $2f_{45}/f_{49} \neq -1$, it approaches -1 for large j, and so a similar combination $\mathcal{E} - \mathcal{E}'$ is involved in that limit. Especially in a protoplanetary system in which the disc and planet(s) have comparable orbital angular momenta and engage in coupled secular dynamics, the eccentricities and inclinations of both disc and companion(s) must be considered simultaneously.

4.2 First-order resonances

In the case of a circular and coplanar system the only operative term in the disturbing function for an interior first-order $j : j-1$ resonance is

$$R = \frac{GM'}{a'} f_{27}\mathcal{E}. \tag{106}$$

This leads to no evolution of z_α or z'_α (which remain zero) but does produce a torque on the disc, which is (according to equation 77)

$$-\frac{\pi j}{\Lambda_0 |D|}\left(\frac{GM'}{a'}\right)^2 f_{27}^2 \frac{\mathrm{d}m}{\mathrm{d}a_0}. \tag{107}$$

Using the fact that $f_{27} = -jb_{1/2}^{(j)}(\alpha) - (\alpha/2)\mathrm{d}b_{1/2}^{(j)}/\mathrm{d}\alpha$, this torque can be shown to agree with equation (13) of Goldreich & Tremaine (1980). The equal and opposite torque on the companion follows from equation (97).

When small eccentricities and inclinations are allowed for we obtain the following dynamics to lowest degree:

$$\begin{aligned}\mathrm{i}\langle\dot{z}_1\rangle = {}& \frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda_0}\right)^{3/2} f_{27}\left\{f\left[2(f_{28} - \tfrac{1}{8}f_{27})\mathcal{E} + 2f_{35}\mathcal{E}'\right] + f^*\left[2(f_{28} - \tfrac{1}{8}f_{27})\mathcal{E} + (f_{32} + \alpha\delta_{j,2})\mathcal{E}'\right]\right\} \\ & -\frac{1}{4}(j-1)\left(\frac{GM'}{a'}\right)^2\left\{\frac{\partial}{\partial\Lambda_0}\left[\left(\frac{2}{\Lambda_0}\right) f f_{27}^2\right] z_1 + \frac{\partial}{\partial\Lambda_0}\left[\left(\frac{2}{\Lambda_0}\right)^{1/2}\left(\frac{2}{\Lambda'_0}\right)^{1/2} f f_{27}(f_{31} - 2\alpha\delta_{j,2})\right] z'_1\right\},\end{aligned} \tag{108}$$

$$\mathrm{i}\langle\dot{z}_2\rangle = \frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda_0}\right)^{3/2} f_{27}\left\{f\left[-\tfrac{1}{8}f_{39}\mathcal{I} + \tfrac{1}{8}f_{40}(2\mathcal{I}' - \mathcal{I})\right] + f^*\left[\tfrac{1}{8}f_{39}(2\mathcal{I}' - \mathcal{I}) - \tfrac{1}{8}f_{40}\mathcal{I}\right]\right\}, \tag{109}$$

$$\begin{aligned}\mathrm{i}\langle\dot{z}'_1\rangle = {}& \frac{1}{\mu}\int\frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda'_0}\right)^{1/2}\left(\frac{2}{\Lambda_0}\right) f_{27}\left\{f\left[f_{29}\mathcal{E}' + (f_{32} + \alpha\delta_{j,2})\mathcal{E}\right] + f^*\left[f_{29}\mathcal{E}' + 2f_{35}\mathcal{E}\right]\right\}\mathrm{d}m \\ & -\frac{1}{\mu}\int\frac{1}{4}(j-1)\left(\frac{GM'}{a'}\right)^2\left\{\frac{\partial}{\partial\Lambda_0}\left[\left(\frac{2}{\Lambda_0}\right)^{1/2}\left(\frac{2}{\Lambda'_0}\right)^{1/2} f f_{27}(f_{31} - 2\alpha\delta_{j,2})\right] z_1 + \frac{\partial}{\partial\Lambda_0}\left[\left(\frac{2}{\Lambda'_0}\right) f (f_{31} - 2\alpha\delta_{j,2})^2\right] z'_1\right\}\mathrm{d}m,\end{aligned} \tag{110}$$

$$\mathrm{i}\langle\dot{z}'_2\rangle = \frac{1}{\mu}\int\frac{1}{4}\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda'_0}\right)^{1/2}\left(\frac{2}{\Lambda_0}\right) f_{27}\left\{f\left[\tfrac{1}{8}f_{39}(2\mathcal{I} - \mathcal{I}') - \tfrac{1}{8}f_{40}\mathcal{I}'\right] + f^*\left[-\tfrac{1}{8}f_{39}\mathcal{I}' + \tfrac{1}{8}f_{40}(2\mathcal{I} - \mathcal{I}')\right]\right\}\mathrm{d}m. \tag{111}$$

Both 'Lindblad' and 'corotation' aspects of the resonance enter here. Regarding the 'corotation' terms (those involving derivatives with respect to Λ_0), we again have precessional behaviour that diverges, this time more strongly, at the resonance, owing to the appearance of $\mathrm{d}f/\mathrm{d}\Lambda_0$. Resolution of the resonant singularity leads to an eccentricity growth rate (e.g. when $\mathcal{E}' = 0$) that changes sign across the location of the resonance. Which sign is obtained for the disc as a whole depends on whether $\mathrm{d}m/\mathrm{d}\Lambda_0$ is greater on one side or the other. The net effect for a narrow resonance is proportional to $\mathrm{d}^2m/\mathrm{d}\Lambda_0^2$, or equivalently $\mathrm{d}(\Sigma/n)/\mathrm{d}a_0$, as is characteristic of corotation resonances.

[11] In his equation the second K_1 should be K_0.

It is notable that $\langle \dot{z}_2 \rangle$ and $\langle \dot{z}_2' \rangle$ do not vanish when the disc and companion have inclined but coplanar orbits ($\mathcal{I} = \mathcal{I}' \neq 0$). Using the relation $f_{39} - f_{40} = 2jf_{27}$ it is possible to show that $\langle \dot{z}_2 \rangle$ and $\langle \dot{z}_2' \rangle$ in this case can be attributed entirely to $\langle \dot{\Lambda} \rangle$ and $\langle \dot{\Lambda}' \rangle$ while $\mathcal{I}$ and $\mathcal{I}'$ remain constant. In general, the rate of change of total AMD is

$$\left(\frac{GM'}{a'}\right)^2 \frac{(j-1)\pi}{2|D|} \frac{\mathrm{d}}{\mathrm{d}a_0} \left(\frac{\mathrm{d}m}{\mathrm{d}\Lambda_0}\right) |f_{27}\mathcal{E} + (f_{31} - 2\alpha\delta_{j,2})\mathcal{E}'|^2. \tag{112}$$

Note that only the 'corotation' terms contribute to this expression. The equivalent result for an exterior first-order resonance is

$$-\left(\frac{GM'}{a}\right)^2 \frac{j\pi}{2|D|} \frac{\mathrm{d}}{\mathrm{d}a_0} \left(\frac{\mathrm{d}m}{\mathrm{d}\Lambda_0}\right) |f_{27}\mathcal{E}' + (f_{31} - \tfrac{1}{2}\alpha^{-2}\delta_{j,2})\mathcal{E}|^2. \tag{113}$$

Using the fact that $f_{31} \approx -f_{27} \approx (j/\pi)[2K_0(2/3) + K_1(2/3)]$ for $j \gg 1$, we find a growth (or decay) rate for $\mathcal{E}'$ that agrees with equation (12) of the analysis of eccentric corotation resonances by Ward (1988) in the case of a circular disc and $e' \ll 1$. Whether the AMD grows or decays depends on the sign of the vortensity gradient. In general, as for second-order resonances, the eccentricities of both disc and companion must be considered simultaneously, as the resonant effect depends on a linear combination of $\mathcal{E}$ and $\mathcal{E}'$, in fact $\mathcal{E} - \mathcal{E}'$ in the limit of large j.

4.3 Zeroth-order resonances

Zeroth-order resonances are coorbital and are active only if a clean gap is not opened by the companion's orbit. In the case of a circular and coplanar system the only operative term in the disturbing function for a zeroth-order resonance is

$$R = \frac{GM'}{a'}(2f_1 - \alpha\delta_{j,1}). \tag{114}$$

This leads to no evolution of z_α or z_α' (which remain zero) but does produce a torque on the disc, which is (according to equation 77)

$$-\frac{\pi j^2}{2|D|}\left(\frac{GM'}{a'}\right)^2 (2f_1 - \alpha\delta_{j,1})^2 \frac{\mathrm{d}}{\mathrm{d}a_0}\left(\frac{\mathrm{d}m}{\mathrm{d}\Lambda_0}\right). \tag{115}$$

Using the fact that $f_1 = \frac{1}{2}b_{1/2}^{(j)}(\alpha)$, this torque can be shown to agree with equation (14) of Goldreich & Tremaine (1980), and the equal and opposite torque on the companion follows from equation (97). Setting $\alpha = 1$ renders the coefficients singular, and this must be resolved in practice by a smoothing process that represents an averaging of the potential over the vertical extent of the disc, which we do not consider here.

When small eccentricities and inclinations are allowed for we obtain the following dynamics to lowest degree:

$$\mathrm{i}\langle \dot{z}_1 \rangle = \frac{1}{4}\left(\frac{GM'}{a'}\right)^2 \left(\frac{2}{\Lambda_0}\right)^{3/2} (2f_2 + \tfrac{1}{2}\alpha\delta_{j,1}) \left\{-f\left[(2f_2 + \tfrac{1}{2}\alpha\delta_{j,1})\mathcal{E} + (\tilde{f}_{10} - \alpha\delta_{j,2})\mathcal{E}'\right] + f^*\left[(2f_2 + \tfrac{1}{2}\alpha\delta_{j,1})\mathcal{E} + f_{10}\mathcal{E}'\right]\right\}, \tag{116}$$

$$\mathrm{i}\langle \dot{z}_2 \rangle = \frac{1}{4}\left(\frac{GM'}{a'}\right)^2 \left(\frac{2}{\Lambda_0}\right)^{3/2} \tfrac{1}{4}(2f_3 + \alpha\delta_{j,1}) \left\{-f\left[\tfrac{1}{4}(2f_3 + \alpha\delta_{j,1})\mathcal{I} + \tfrac{1}{4}(\tilde{f}_{14} - 2\alpha\delta_{j,1})\mathcal{I}'\right] + f^*\left[\tfrac{1}{4}(2f_3 + \alpha\delta_{j,1})\mathcal{I} + \tfrac{1}{4}f_{14}\mathcal{I}'\right]\right\}, \tag{117}$$

$$\begin{aligned}\mathrm{i}\langle \dot{z}_1' \rangle = \frac{1}{\mu}\int \frac{1}{4}\left(\frac{GM'}{a'}\right)^2 \left(\frac{2}{\Lambda_0'}\right)^{1/2} \left(\frac{2}{\Lambda_0}\right) \Big\{ & -f(\tilde{f}_{10} - \alpha\delta_{j,2})\left[(2f_2 + \tfrac{1}{2}\alpha\delta_{j,1})\mathcal{E} + (\tilde{f}_{10} - \alpha\delta_{j,2})\mathcal{E}'\right] \\ & + f^* f_{10}\left[(2f_2 + \tfrac{1}{2}\alpha\delta_{j,1})\mathcal{E} + f_{10}\mathcal{E}'\right]\Big\}\,\mathrm{d}m,\end{aligned} \tag{118}$$

$$\begin{aligned}\mathrm{i}\langle \dot{z}_2' \rangle = \frac{1}{\mu}\int \frac{1}{4}\left(\frac{GM'}{a'}\right)^2 \left(\frac{2}{\Lambda_0'}\right)^{1/2} \left(\frac{2}{\Lambda_0}\right) \Big\{ & -f\tfrac{1}{4}(2\tilde{f}_{14} - 2\alpha\delta_{j,1})\left[\tfrac{1}{4}(2f_3 + \alpha\delta_{j,1})\mathcal{I} + \tfrac{1}{4}(\tilde{f}_{14} - 2\alpha\delta_{j,1})\mathcal{I}'\right] \\ & + f^*\tfrac{1}{4}f_{14}\left[\tfrac{1}{4}(2f_3 + \alpha\delta_{j,1})\mathcal{I} + \tfrac{1}{4}f_{14}\mathcal{I}'\right]\Big\}\,\mathrm{d}m.\end{aligned} \tag{119}$$

For brevity we omit 'corotation' effects here and focus on the 'Lindblad' terms. The rate of change of total AMD is

$$\begin{aligned}-\left(\frac{GM'}{a'}\right)^2 \frac{\pi}{\Lambda_0|D|}\frac{\mathrm{d}m}{\mathrm{d}a_0}\Big[& |(2f_2 + \tfrac{1}{2}\alpha\delta_{j,1})\mathcal{E} + (\tilde{f}_{10} - \alpha\delta_{j,2})\mathcal{E}'|^2 + |(2f_2 + \tfrac{1}{2}\alpha\delta_{j,1})\mathcal{E} + f_{10}\mathcal{E}'|^2 \\ & + |\tfrac{1}{4}(2f_3 + \alpha\delta_{j,1})\mathcal{I} + \tfrac{1}{4}(\tilde{f}_{14} - 2\alpha\delta_{j,1})\mathcal{I}'|^2 + |\tfrac{1}{4}(2f_3 + \alpha\delta_{j,1})\mathcal{I} + \tfrac{1}{4}f_{14}\mathcal{I}'|^2\Big],\end{aligned} \tag{120}$$

again omitting 'corotation' terms. Although these expressions cannot be consistently evaluated without a softening procedure and the further considerations of Ward (1988), we see that the effect of the coorbital 'Lindblad' terms is to damp the eccentricity and inclination of the disc or companion if the other is fixed at zero eccentricity and inclination. Generally, however, the effect of the resonance depends on linear combinations of $\mathcal{E}$ and $\mathcal{E}'$, and of $\mathcal{I}$ and $\mathcal{I}'$, which are again $\mathcal{E}-\mathcal{E}'$ and $\mathcal{I}-\mathcal{I}'$ in the limit of large j. In this sense it is the (complex) differences in eccentricities and inclinations of the disc and companion that are being damped, so these terms attempt to equalize both e and ϖ (or i and Ω) of the disc and companion.

Again, it is notable that $\langle\dot z_2\rangle$ and $\langle\dot z_2'\rangle$ do not vanish when the disc and companion have inclined but coplanar orbits ($\mathcal{I}=\mathcal{I}'\neq 0$). Using the relations $2f_3+f_{14}+\alpha\delta_{j,1}=-2f_3-\tilde f_{14}+\alpha\delta_{j,1}=-j(2f_1-\alpha\delta_{j,1})$ it is possible to show that $\langle\dot z_2\rangle$ and $\langle\dot z_2'\rangle$ in this case can be attributed entirely to $\langle\dot\Lambda\rangle$ and $\langle\dot\Lambda'\rangle$ while $\mathcal{I}$ and $\mathcal{I}'$ remain constant.

4.4 Effects beyond the first degree in eccentricity

For a companion on a circular orbit (e.g. in a close binary star) and a coplanar disc, the disturbing function for an interior second-order resonance is

$$R=\frac{GM'}{a'}\left[f_{45}\mathcal{E}^2+(f_{46}-\tfrac14 f_{45})|\mathcal{E}|^2\mathcal{E}^2+O(\mathcal{E}^6)\right]. \tag{121}$$

In this case

$$\begin{aligned}\mathrm{i}\langle\dot z_1\rangle=&\frac14\left(\frac{GM'}{a'}\right)^2\left(\frac{2}{\Lambda_0}\right)^2 f_{45}\left[4ff_{45}+6(3f+f^*)(f_{46}-\tfrac14 f_{45})|\mathcal{E}|^2\right]z_1-\frac14(j-2)\left(\frac{GM'}{a'}\right)^2\frac{\partial}{\partial\Lambda_0}\left[\left(\frac{2}{\Lambda_0}\right)^2 2ff_{45}^2\right]|z_1|^2z_1\\&+O(z^5).\end{aligned} \tag{122}$$

The effect on the AMD of the disc is

$$\begin{aligned}\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\int|z_\beta|^2\,\mathrm{d}m\right\rangle=&\left(\frac{GM'}{a'}\right)^2\frac{\pi}{\Lambda_0|D|}\left\{\frac{\mathrm{d}m}{\mathrm{d}a_0}f_{45}\left[4f_{45}|\mathcal{E}|^2+12(f_{46}-\tfrac14 f_{45})|\mathcal{E}|^4\right]+(j-2)\Lambda_0\frac{\mathrm{d}}{\mathrm{d}a_0}\left(\frac{\mathrm{d}m}{\mathrm{d}\Lambda_0}\right)f_{45}^2|\mathcal{E}|^4\right\}\\&+O(z^6),\end{aligned} \tag{123}$$

evaluated at the resonance.

As a specific example, for the 3:1 resonance we have $f_{45}=0.5988$ and $f_{46}=-0.1936$. The terms in braces in the above equation are therefore

$$1.434e^2\frac{\mathrm{d}m}{\mathrm{d}a_0}\left[1-1.470e^2-\tfrac12\beta e^2+O(e^4)\right], \tag{124}$$

where β is the logarithmic derivative of vortensity with respect to a. This result implies that the eccentricity growth rate is substantially reduced when e is not small, providing a possible means to limit the growth of eccentricity in superhump systems.

To evaluate consistently the evolution to higher degrees in eccentricity and inclination, further resonances must be taken into account. For an nth-order resonance ($n\geq 2$) we have $R\propto\mathcal{E}^n$, leading to a growth of eccentricity in the disc with $\dot{\mathcal{E}}\propto|\mathcal{E}|^{2n-4}\mathcal{E}$.

5 SUMMARY AND CONCLUSIONS

We have presented a new and general analysis of mean-motion resonances between a Keplerian disc and an orbiting companion. The emphasis of this treatment is to provide a systematic method to calculate the rates of change of eccentricity and inclination variables of the disc and companion associated with resonances of various orders.

Near a mean-motion resonance, terms in the disturbing function corresponding to the appropriate commensurability produce an oscillation of the orbital elements of the disc and companion at first order in the perturbation. At second order, a secular effect is obtained in the form of a precessional dynamics that diverges as the resonance is approached. When the divergence is resolved by taking into account the (dissipative) collective effects of the disc, a finite precession rate is obtained, together with an irreversible evolution, such as a growth or decay of eccentricity or inclination.

Our work differs from earlier methods in which the perturbing potential is decomposed into rigidly rotating components and the eccentricity evolution is deduced indirectly from resonant torques (Goldreich & Tremaine 1980; Ward 1988). Such methods do not apply to general satellite–disc interactions in which both the disc and companion have eccentric and/or inclined orbits. Where appropriate, however, we obtain full agreement with this earlier work. By using the classical disturbing function, we benefit from the ready availability of expansions to high degrees in eccentricity and inclination.

After considerable thought, we recommend the use of complex canonical Poincaré variables, as defined in Section 2.3, to describe the eccentricity and inclination dynamics. These allow the most compact expression of the dynamical equations and the disturbing function. They are also directly related to the angular momentum deficit (AMD), which is a quantity of fundamental importance, being a positive-definite measure of the departure of the system from circularity and coplanarity that is conserved in first-order secular interactions. By concentrating on the evolution of the system's AMD we can separate the effects that lead to irreversible growth or decay of eccentricity and inclination from those that are merely precessional in character. Moreover, the use of complex variables leads, in the case of small eccentricities and inclinations, to linear evolutionary equations that would be extremely cumbersome in any other representation.

This work does not address certain refinements that are necessary for a more complete description of resonant interactions. The important omitted effects are nonlinearity (in particular the saturation of corotation resonances), torque cutoff effects, resonant shifts and vertical averaging. It is likely that all of these could be addressed within the present framework.

The equations derived here do not stand alone but should be combined, in future work, with the complex one-dimensional partial differential equations describing the secular evolution of eccentricity and inclination in the disc. In simple cases such a programme has already been carried out (Lubow & Ogilvie 2001; Goodchild & Ogilvie 2006). For small eccentricity and inclination the outcome is a set of secular normal modes, being a continuum analogue of the Laplace–Lagrange secular theory, which describe rigidly precessing configurations of disc(s) and companion(s). The contribution of mean-motion resonances allows these modes to grow (or decay), and any growth must be offset against the viscous damping of the mode within the disc. Coupled eccentric modes have also been discussed by Papaloizou (2002).

Previous discussions of eccentricity growth in protoplanetary systems, for example, have presented an interesting but incomplete picture. Goldreich & Tremaine (1980), Ward (1988) and Goldreich & Sari (2003) discuss the close competition between eccentric Lindblad resonances and eccentric corotation resonances. This is equivalent to comparing equations (104) and (112) with the complex eccentricity $\mathcal{E}$ of the disc set to zero (and $\mathcal{I} = \mathcal{I}' = 0$). Since the disc becomes eccentric through secular (and possibly resonant) interactions, the relative weighting of all these resonances is adjusted in a way that depends on the eccentricity distribution within the disc. A self-consistent solution of the coupled secular dynamics, as carried out by Lubow & Ogilvie (2001) in the case of inclination, is therefore required to address the issue. It remains to be seen whether protoplanetary systems support growing eccentric modes in the presence of viscous damping (see also Latter & Ogilvie 2006).

Another application of this work is to close binary stars, where accretion discs can sometimes become eccentric through the action of the 3:1 resonance. This work suggests various elaborations of the description by Lubow (1991). In addition to eccentricity growth, the 3:1 resonance provides an second-order precession that may affect the shape of the eccentric mode and alter the global precession rate of the disc. The growth rate is weakened when the eccentricity is no longer small, leading to a possible saturation mechanism for the superhump instability. Finally, higher-order resonances such as 4 : 1 provide eccentricity growth of the form $\dot{\mathcal{E}} \propto |\mathcal{E}|^2\mathcal{E}$ which, although not leading to a linear instability, might conceivably support a global eccentric mode in binary stars of larger mass ratio, where the disc is too small to contain the 3:1 resonance.

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APPENDIX A: EXPANSION OF THE DISTURBING FUNCTION

In this section the disturbing function is given for various commensurabilities, correct to fourth degree in eccentricity and inclination. The coefficients f_i are defined in Appendix B of Murray & Dermott (1999).

A1 Secular terms

$$\begin{aligned}
R_{0,0} = \frac{GM'}{a'}\Big\{ & f_1 + f_2\left[|\mathcal{E}|^2 + |\mathcal{E}'|^2 - |\mathcal{I}-\mathcal{I}'|^2 + \tfrac{1}{4}\left(|\mathcal{E}|^2+|\mathcal{E}'|^2\right)\left(\mathcal{I}\mathcal{I}'^* + \mathcal{I}^*\mathcal{I}'\right) - \tfrac{1}{2}\left(|\mathcal{E}|^2|\mathcal{I}|^2 + |\mathcal{E}'|^2|\mathcal{I}'|^2\right)\right] \\
& + (f_4 - \tfrac{1}{4}f_2)|\mathcal{E}|^4 + f_5\left[|\mathcal{E}|^2|\mathcal{E}'|^2 - \left(|\mathcal{E}|^2+|\mathcal{E}'|^2\right)|\mathcal{I}-\mathcal{I}'|^2\right] + (f_6 - \tfrac{1}{4}f_2)|\mathcal{E}'|^4 + \tfrac{1}{2}f_{17}\left(\mathcal{E}^2\mathcal{E}'^{*2} + \mathcal{E}^{*2}\mathcal{E}'^2\right) \\
& + \tfrac{1}{2}\left(\mathcal{E}\mathcal{E}'^* + \mathcal{E}^*\mathcal{E}'\right)\left[f_{10} + (f_{11} - \tfrac{1}{8}f_{10})|\mathcal{E}|^2 + (f_{12} - \tfrac{1}{8}f_{10})|\mathcal{E}'|^2 + \tfrac{1}{4}f_{13}|\mathcal{I}-\mathcal{I}'|^2\right] + \tfrac{1}{16}f_8|\mathcal{I}-\mathcal{I}'|^4 \\
& + \tfrac{1}{16}(f_8 - 4f_5)\left[\left(|\mathcal{I}|^2+|\mathcal{I}'|^2\right)\left(\mathcal{I}\mathcal{I}'^* + \mathcal{I}^*\mathcal{I}'\right) - 4|\mathcal{I}|^2|\mathcal{I}'|^2\right] + \tfrac{1}{8}f_{18}\left[\mathcal{E}^2(\mathcal{I}^* - \mathcal{I}'^*)^2 + \mathcal{E}^{*2}(\mathcal{I}-\mathcal{I}')^2\right] \\
& + \tfrac{1}{8}f_{19}\left[\mathcal{E}\mathcal{E}'(\mathcal{I}^* - \mathcal{I}'^*)^2 + \mathcal{E}^*\mathcal{E}'^*(\mathcal{I}-\mathcal{I}')^2\right] + \tfrac{1}{8}f_{20}\left[\mathcal{E}'^2(\mathcal{I}^* - \mathcal{I}'^*)^2 + \mathcal{E}'^{*2}(\mathcal{I}-\mathcal{I}')^2\right] \\
& - \tfrac{1}{8}(2f_{19} + f_{13})\left(\mathcal{E}\mathcal{E}'^* - \mathcal{E}^*\mathcal{E}'\right)\left(\mathcal{I}\mathcal{I}'^* - \mathcal{I}^*\mathcal{I}'\right)\Big\},
\end{aligned} \tag{A1}$$

using coefficients f_i as defined in Table B.1 of Murray & Dermott (1999) with $j = 0$. This expression can be shown to agree with that given by Laskar & Robutel (1995). Note that $R_{0,0}$ is real, and contributes negatively to the secular Hamiltonian.

A2 Zeroth-order resonances

Co-orbital $j:j$ resonance with $j \geq 1$, for which $k = j$, $k' = -j$, $\dot\phi_0 = jn - jn'$ and $\alpha = 1$:

$$\begin{aligned}
R = \frac{GM'}{a'}\Big\{ & (2f_1 - \alpha\delta_{j,1}) + (2f_2 + \tfrac{1}{2}\alpha\delta_{j,1})\left(|\mathcal{E}|^2 + |\mathcal{E}'|^2\right) + (2f_4 - \tfrac{1}{2}f_2 - \tfrac{7}{64}\alpha\delta_{j,1})|\mathcal{E}|^4 + (2f_5 - \tfrac{1}{4}\alpha\delta_{j,1})|\mathcal{E}|^2|\mathcal{E}'|^2 \\
& + (2f_6 - \tfrac{1}{2}f_2 - \tfrac{7}{64}\alpha\delta_{j,1})|\mathcal{E}'|^4 + f_{17}\mathcal{E}^2\mathcal{E}'^{*2} + (\tilde f_{17} - \tfrac{1}{64}\alpha\delta_{j,1} - \tfrac{81}{64}\alpha\delta_{j,3})\mathcal{E}^{*2}\mathcal{E}'^2 \\
& + \mathcal{E}\mathcal{E}'^*\left[f_{10} + (f_{11} - \tfrac{1}{8}f_{10})|\mathcal{E}|^2 + (f_{12} - \tfrac{1}{8}f_{10})|\mathcal{E}'|^2 + \tfrac{1}{4}f_{13}\left(|\mathcal{I}|^2 + |\mathcal{I}'|^2\right) + \tfrac{1}{4}f_{22}\mathcal{I}^*\mathcal{I}' + \tfrac{1}{4}f_{23}\mathcal{I}\mathcal{I}'^*\right] \\
& + \mathcal{E}^*\mathcal{E}'\Big[(\tilde f_{10} - \alpha\delta_{j,2}) + (\tilde f_{11} - \tfrac{1}{8}\tilde f_{10} + \tfrac{7}{8}\alpha\delta_{j,2})|\mathcal{E}|^2 + (\tilde f_{12} - \tfrac{1}{8}\tilde f_{10} + \tfrac{7}{8}\alpha\delta_{j,2})|\mathcal{E}'|^2 + \tfrac{1}{4}(\tilde f_{13} + \alpha\delta_{j,2})\left(|\mathcal{I}|^2 + |\mathcal{I}'|^2\right) \\
& \quad + \tfrac{1}{4}\tilde f_{22}\mathcal{I}\mathcal{I}'^* + \tfrac{1}{4}(\tilde f_{23} - 2\alpha\delta_{j,2})\mathcal{I}^*\mathcal{I}'\Big] + \tfrac{1}{4}(2f_3 + \alpha\delta_{j,1})\left(|\mathcal{I}|^2 + |\mathcal{I}'|^2\right) + \tfrac{1}{8}f_8\left(|\mathcal{I}|^4 + |\mathcal{I}'|^4\right) + \tfrac{1}{16}(2f_9 - \alpha\delta_{j,1})|\mathcal{I}|^2|\mathcal{I}'|^2 \\
& + \tfrac{1}{16}f_{26}\mathcal{I}^2\mathcal{I}'^{*2} + \tfrac{1}{16}(\tilde f_{26} - \alpha\delta_{j,1})\mathcal{I}^{*2}\mathcal{I}'^2 + \tfrac{1}{4}\mathcal{I}\mathcal{I}'^*\left[f_{14} + (f_{15} + \tfrac{1}{4}f_{14})\left(|\mathcal{E}|^2 + |\mathcal{E}'|^2\right) + \tfrac{1}{4}f_{16}\left(|\mathcal{I}|^2 + |\mathcal{I}'|^2\right)\right] \\
& + \tfrac{1}{4}\mathcal{I}^*\mathcal{I}'\left[(\tilde f_{14} - 2\alpha\delta_{j,1}) + (\tilde f_{15} + \tfrac{1}{4}\tilde f_{14} + \tfrac{1}{2}\alpha\delta_{j,1})\left(|\mathcal{E}|^2 + |\mathcal{E}'|^2\right) + \tfrac{1}{4}(\tilde f_{16} + \alpha\delta_{j,1})\left(|\mathcal{I}|^2 + |\mathcal{I}'|^2\right)\right] \\
& + \tfrac{1}{4}(f_3 + 2f_7)\left(|\mathcal{E}|^2|\mathcal{I}|^2 + |\mathcal{E}'|^2|\mathcal{I}'|^2\right) + \tfrac{1}{4}(2f_7 - \tfrac{1}{2}\alpha\delta_{j,1})\left(|\mathcal{E}|^2|\mathcal{I}'|^2 + |\mathcal{E}'|^2|\mathcal{I}|^2\right) \\
& + \tfrac{1}{4}(\mathcal{I}^2 + \mathcal{I}'^2)\left(f_{18}\mathcal{E}^{*2} + f_{19}\mathcal{E}^*\mathcal{E}'^* + f_{20}\mathcal{E}'^{*2}\right) + \tfrac{1}{4}(\mathcal{I}^{*2} + \mathcal{I}'^{*2})\left(\tilde f_{18}\mathcal{E}^2 + \tilde f_{19}\mathcal{E}\mathcal{E}' + \tilde f_{20}\mathcal{E}'^2\right) \\
& + \tfrac{1}{4}\mathcal{I}\mathcal{I}'\left(f_{21}\mathcal{E}^{*2} + f_{24}\mathcal{E}^*\mathcal{E}'^* + f_{25}\mathcal{E}'^{*2}\right) + \tfrac{1}{4}\mathcal{I}^*\mathcal{I}'^*\left(\tilde f_{21}\mathcal{E}^2 + \tilde f_{24}\mathcal{E}\mathcal{E}' + \tilde f_{25}\mathcal{E}'^2\right) \\
& - \tfrac{1}{32}\alpha\delta_{j,1}\left[\mathcal{E}^{*2}(\mathcal{I}-\mathcal{I}')^2 + \mathcal{E}'^2(\mathcal{I}^* - \mathcal{I}'^*)^2\right]\Big\}.
\end{aligned} \tag{A2}$$

Here $\tilde f_i$ denotes the coefficient f_i in which j is replaced with $-j$. Setting $\alpha = 1$ renders the coefficients singular, and this must be resolved in practice by a smoothing process that represents averaging over the vertical extent of the disc.

A3 First-order resonances

Interior $j : j-1$: resonance with $j \geq 2$, for which $k = j-1$, $k' = -j$, $\dot\phi_0 = (j-1)n - jn'$ and $\alpha = a/a' = [(j-1)/j]^{2/3}$:

$$\begin{aligned}
R = \frac{GM'}{a'}\Big\{ & f_{27}\mathcal{E} + (f_{31} - 2\alpha\delta_{j,2})\mathcal{E}' + (f_{28} - \tfrac{1}{8}f_{27})|\mathcal{E}|^2\mathcal{E} + f_{29}|\mathcal{E}'|^2\mathcal{E} + (f_{32} + \alpha\delta_{j,2})|\mathcal{E}|^2\mathcal{E}' + (f_{33} - \tfrac{1}{8}f_{31} + \tfrac{3}{2}\alpha\delta_{j,2})|\mathcal{E}'|^2\mathcal{E}' \\
& + f_{35}\mathcal{E}^2\mathcal{E}'^* + (f_{36} - \tfrac{27}{16}\alpha\delta_{j,3})\mathcal{E}'^2\mathcal{E}^* + \tfrac{1}{8}[f_{39}\mathcal{E} + (f_{42} - 4\alpha\delta_{j,2})\mathcal{E}']\left[\mathcal{I}^*(\mathcal{I}' - \mathcal{I}) - \mathcal{I}'(\mathcal{I}'^* - \mathcal{I}^*)\right] \\
& + \tfrac{1}{8}(f_{40}\mathcal{E} + f_{43}\mathcal{E}')\left[\mathcal{I}(\mathcal{I}'^* - \mathcal{I}^*) - \mathcal{I}'^*(\mathcal{I}' - \mathcal{I})\right] + \frac{1}{4}(f_{37}\mathcal{E}^* + f_{38}\mathcal{E}'^*)(\mathcal{I} - \mathcal{I}')^2\Big\}.
\end{aligned} \tag{A3}$$

Exterior $j-1:j$ resonance with $j\geq 2$, for which $k=-j$, $k'=j-1$, $\dot{\phi}_0=-jn+(j-1)n'$ and $\alpha=a'/a=[(j-1)/j]^{2/3}$:

$$\begin{aligned}R=\frac{GM'}{a}\big\{&f_{27}\mathcal{E}'+(f_{31}-\tfrac{1}{2}\alpha^{-2}\delta_{j,2})\mathcal{E}+(f_{28}-\tfrac{1}{8}f_{27})|\mathcal{E}'|^2\mathcal{E}'+f_{29}|\mathcal{E}|^2\mathcal{E}'+(f_{32}+\tfrac{1}{4}\alpha^{-2}\delta_{j,2})|\mathcal{E}'|^2\mathcal{E}\\&+(f_{33}-\tfrac{1}{8}f_{31}+\tfrac{7}{16}\alpha^{-2}\delta_{j,2})|\mathcal{E}|^2\mathcal{E}+f_{35}\mathcal{E}'^2\mathcal{E}^*+(f_{36}-\tfrac{3}{4}\alpha^{-2}\delta_{j,3})\mathcal{E}^2\mathcal{E}'^*\\&+\tfrac{1}{8}[f_{39}\mathcal{E}'+(f_{42}-\alpha^{-2}\delta_{j,2})\mathcal{E}]\left[\mathcal{I}'^*(\mathcal{I}-\mathcal{I}')-\mathcal{I}(\mathcal{I}^*-\mathcal{I}'^*)\right]\\&+\tfrac{1}{8}(f_{40}\mathcal{E}'+f_{43}\mathcal{E})\left[\mathcal{I}'(\mathcal{I}^*-\mathcal{I}'^*)-\mathcal{I}^*(\mathcal{I}-\mathcal{I}')\right]+\frac{1}{4}(f_{37}\mathcal{E}'^*+f_{38}\mathcal{E}^*)(\mathcal{I}'-\mathcal{I})^2\big\}.\end{aligned}\tag{A4}$$

A4 Second-order resonances

Interior $j:j-2$ resonance with $j\geq 3$, for which $k=j-2$, $k'=-j$, $\dot{\phi}_0=(j-2)n-jn'$ and $\alpha=a/a'=[(j-2)/j]^{2/3}$:

$$\begin{aligned}R=\frac{GM'}{a'}\big\{&f_{45}\mathcal{E}^2+f_{49}\mathcal{E}\mathcal{E}'+(f_{53}-\tfrac{27}{8}\alpha\delta_{j,3})\mathcal{E}'^2+(f_{46}-\tfrac{1}{4}f_{45})|\mathcal{E}|^2\mathcal{E}^2+f_{47}|\mathcal{E}'|^2\mathcal{E}^2+(f_{50}-\tfrac{1}{8}f_{49})|\mathcal{E}|^2\mathcal{E}\mathcal{E}'\\&+(f_{51}-\tfrac{1}{8}f_{49})|\mathcal{E}'|^2\mathcal{E}\mathcal{E}'+(f_{54}+\tfrac{27}{16}\alpha\delta_{j,3})|\mathcal{E}|^2\mathcal{E}'^2+(f_{55}-\tfrac{1}{4}f_{53}+\tfrac{135}{32}\alpha\delta_{j,3})|\mathcal{E}'|^2\mathcal{E}'^2+f_{68}\mathcal{E}^3\mathcal{E}'^*+(f_{69}-\tfrac{8}{3}\alpha\delta_{j,4})\mathcal{E}^*\mathcal{E}'^3\\&+\tfrac{1}{4}f_{57}(\mathcal{I}-\mathcal{I}')^2+\tfrac{1}{32}f_{80}[\mathcal{I}^3(\mathcal{I}'^*-\mathcal{I}^*)+\mathcal{I}'^*(\mathcal{I}^3-\mathcal{I}'^3)]+\tfrac{1}{32}f_{81}[\mathcal{I}'^3(\mathcal{I}^*-\mathcal{I}'^*)+\mathcal{I}^*(\mathcal{I}'^3-\mathcal{I}^3)]\\&+\tfrac{1}{64}f_{61}[\mathcal{I}^2\mathcal{I}'(\mathcal{I}'^*-\mathcal{I}^*)-3\mathcal{I}|\mathcal{I}'|^2(\mathcal{I}'-\mathcal{I})]+\tfrac{1}{64}f_{67}[\mathcal{I}'^2\mathcal{I}(\mathcal{I}^*-\mathcal{I}'^*)-3\mathcal{I}'|\mathcal{I}|^2(\mathcal{I}-\mathcal{I}')]\\&+\tfrac{1}{8}f_{57}(\mathcal{I}-\mathcal{I}')(|\mathcal{E}|^2\mathcal{I}-|\mathcal{E}'|^2\mathcal{I}')+\tfrac{1}{8}[f_{72}\mathcal{E}^2+f_{74}\mathcal{E}\mathcal{E}'+(f_{78}-\tfrac{27}{4}\alpha\delta_{j,3})\mathcal{E}'^2]\left[\mathcal{I}^*(\mathcal{I}'-\mathcal{I})-\mathcal{I}'(\mathcal{I}'^*-\mathcal{I}^*)\right]\\&+\tfrac{1}{8}(f_{73}\mathcal{E}^2+f_{75}\mathcal{E}\mathcal{E}'+f_{79}\mathcal{E}'^2)\left[\mathcal{I}(\mathcal{I}'^*-\mathcal{I}^*)-\mathcal{I}'^*(\mathcal{I}'-\mathcal{I})\right]+\tfrac{1}{4}(f_{58}|\mathcal{E}|^2+f_{59}|\mathcal{E}'|^2+f_{70}\mathcal{E}^*\mathcal{E}'+f_{71}\mathcal{E}\mathcal{E}'^*)(\mathcal{I}-\mathcal{I}')^2\big\}.\end{aligned}\tag{A5}$$

Exterior $j-2:j$ resonance with $j\geq 3$, for which $k=-j$, $k'=j-2$, $\dot{\phi}_0=-jn+(j-2)n'$ and $\alpha=a'/a=[(j-2)/j]^{2/3}$:

$$\begin{aligned}R=\frac{GM'}{a}\big\{&f_{45}\mathcal{E}'^2+f_{49}\mathcal{E}'\mathcal{E}+(f_{53}-\tfrac{3}{8}\alpha^{-2}\delta_{j,3})\mathcal{E}^2+(f_{46}-\tfrac{1}{4}f_{45})|\mathcal{E}'|^2\mathcal{E}'^2+f_{47}|\mathcal{E}|^2\mathcal{E}'^2+(f_{50}-\tfrac{1}{8}f_{49})|\mathcal{E}'|^2\mathcal{E}'\mathcal{E}\\&+(f_{54}+\tfrac{3}{16}\alpha^{-2}\delta_{j,3})|\mathcal{E}'|^2\mathcal{E}^2+(f_{55}-\tfrac{1}{4}f_{53}+\tfrac{15}{32}\alpha^{-2}\delta_{j,3})|\mathcal{E}|^2\mathcal{E}^2+f_{68}\mathcal{E}'^3\mathcal{E}^*+(f_{69}-\tfrac{2}{3}\alpha^{-2}\delta_{j,4})\mathcal{E}'^*\mathcal{E}^3\\&+\tfrac{1}{4}f_{57}(\mathcal{I}'-\mathcal{I})^2+\tfrac{1}{32}f_{80}[\mathcal{I}'^3(\mathcal{I}^*-\mathcal{I}'^*)+\mathcal{I}^*(\mathcal{I}'^3-\mathcal{I}^3)]+\tfrac{1}{32}f_{81}[\mathcal{I}^3(\mathcal{I}'^*-\mathcal{I}^*)+\mathcal{I}'^*(\mathcal{I}^3-\mathcal{I}'^3)]\\&+\tfrac{1}{64}f_{61}[\mathcal{I}'^2\mathcal{I}(\mathcal{I}^*-\mathcal{I}'^*)-3\mathcal{I}'|\mathcal{I}|^2(\mathcal{I}-\mathcal{I}')]+\tfrac{1}{64}f_{67}[\mathcal{I}^2\mathcal{I}'(\mathcal{I}'^*-\mathcal{I}^*)-3\mathcal{I}|\mathcal{I}'|^2(\mathcal{I}'-\mathcal{I})]\\&+\tfrac{1}{8}f_{57}(\mathcal{I}'-\mathcal{I})(|\mathcal{E}'|^2\mathcal{I}'-|\mathcal{E}|^2\mathcal{I})+\tfrac{1}{8}[f_{72}\mathcal{E}'^2+f_{74}\mathcal{E}'\mathcal{E}+(f_{78}-\tfrac{3}{4}\alpha^{-2}\delta_{j,3})\mathcal{E}^2]\left[\mathcal{I}'^*(\mathcal{I}-\mathcal{I}')-\mathcal{I}(\mathcal{I}^*-\mathcal{I}'^*)\right]\\&+\tfrac{1}{8}(f_{73}\mathcal{E}'^2+f_{75}\mathcal{E}'\mathcal{E}+f_{79}\mathcal{E}^2)\left[\mathcal{I}'(\mathcal{I}^*-\mathcal{I}'^*)-\mathcal{I}^*(\mathcal{I}-\mathcal{I}')\right]+\tfrac{1}{4}(f_{58}|\mathcal{E}'|^2+f_{59}|\mathcal{E}|^2+f_{70}\mathcal{E}'^*\mathcal{E}+f_{71}\mathcal{E}'\mathcal{E}^*)(\mathcal{I}'-\mathcal{I})^2\big\}.\end{aligned}\tag{A6}$$

A5 Third-order resonances

Interior $j:j-3$ resonance with $j\geq 4$, for which $k=j-3$, $k'=-j$, $\dot{\phi}_0=(j-3)n-jn'$ and $\alpha=a/a'=[(j-3)/j]^{2/3}$:

$$R=\frac{GM'}{a'}\left[f_{82}\mathcal{E}^3+f_{83}\mathcal{E}^2\mathcal{E}'+f_{84}\mathcal{E}\mathcal{E}'^2+(f_{85}-\tfrac{16}{3}\alpha\delta_{j,4})\mathcal{E}'^3+\tfrac{1}{4}(f_{86}\mathcal{E}+f_{87}\mathcal{E}')(\mathcal{I}-\mathcal{I}')^2\right].\tag{A7}$$

Exterior $j-3:j$ resonance with $j\geq 4$, for which $k=-j$, $k'=j-3$, $\dot{\phi}_0=-jn+(j-3)n'$ and $\alpha=a'/a=[(j-3)/j]^{2/3}$:

$$R=\frac{GM'}{a}\left[f_{82}\mathcal{E}'^3+f_{83}\mathcal{E}'^2\mathcal{E}+f_{84}\mathcal{E}'\mathcal{E}^2+(f_{85}-\tfrac{1}{3}\alpha^{-2}\delta_{j,4})\mathcal{E}^3+\tfrac{1}{4}(f_{86}\mathcal{E}'+f_{87}\mathcal{E})(\mathcal{I}'-\mathcal{I})^2\right].\tag{A8}$$